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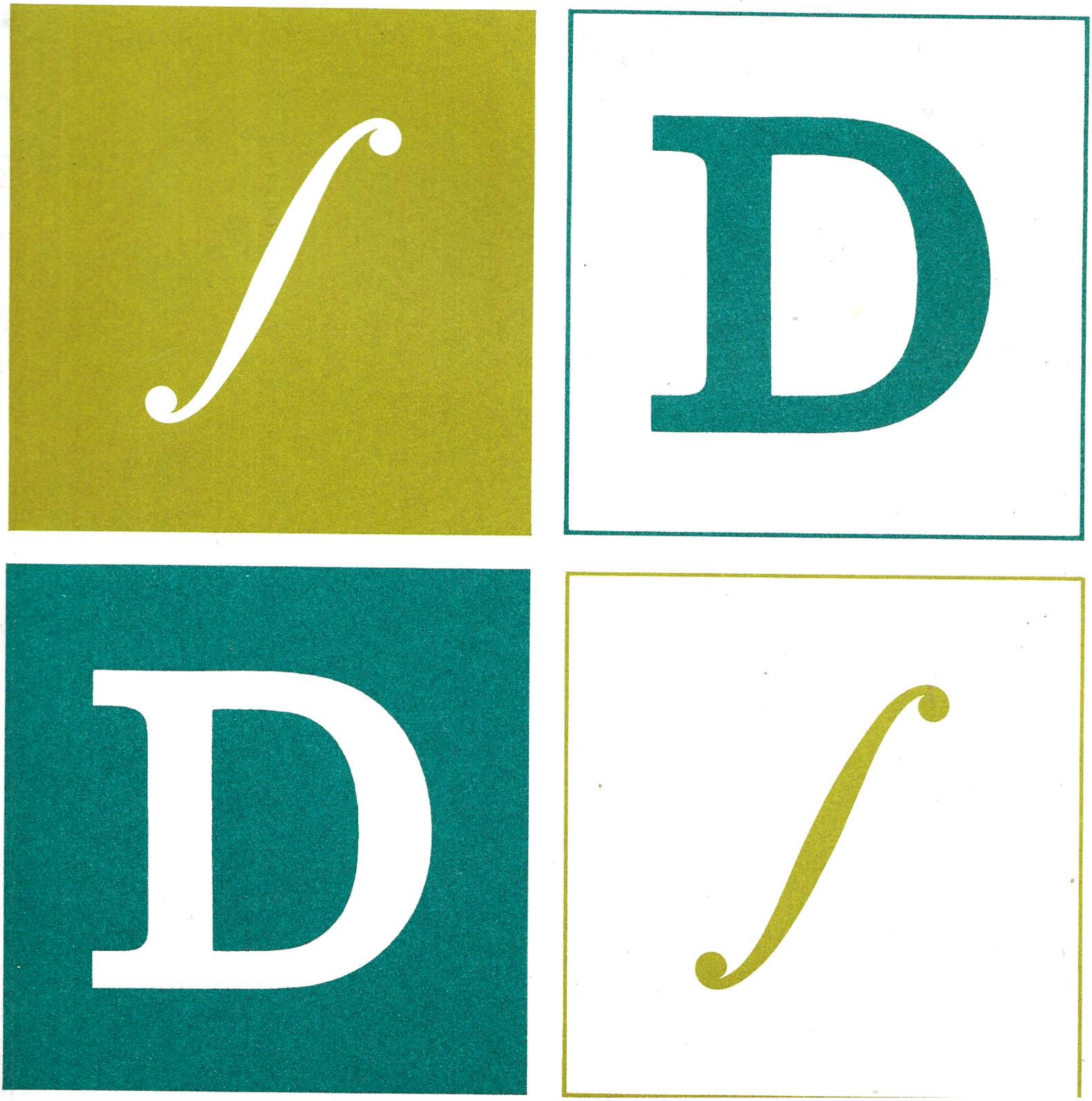
Mathematics Foundation Course Unit 13

Integration II



The Open University

Mathematics Foundation Course Unit 13

INTEGRATION II

Prepared by the Mathematics Foundation Course Team

Correspondence Text 13

The Open University Press

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Contents

Objectives

The primary aim of this unit is to explain the Fundamental Theorem of Calculus and to use it to derive routine procedures for obtaining the primitive functions of any function we may need to integrate (provided a primitive function which can be expressed in terms of known functions exists).

After working through this unit, you should be able to:

(i) explain the meaning of the Fundamental Theorem in your own words; discuss the main features of the proof, and decide whether or not a given function satisfies the conditions necessary for the validity of the proof;

(ii) find primitive functions and definite integrals of simple functions using the table of primitives given in section 13.3.1;

(iii) apply the method of integration by parts to suitable given integrals;

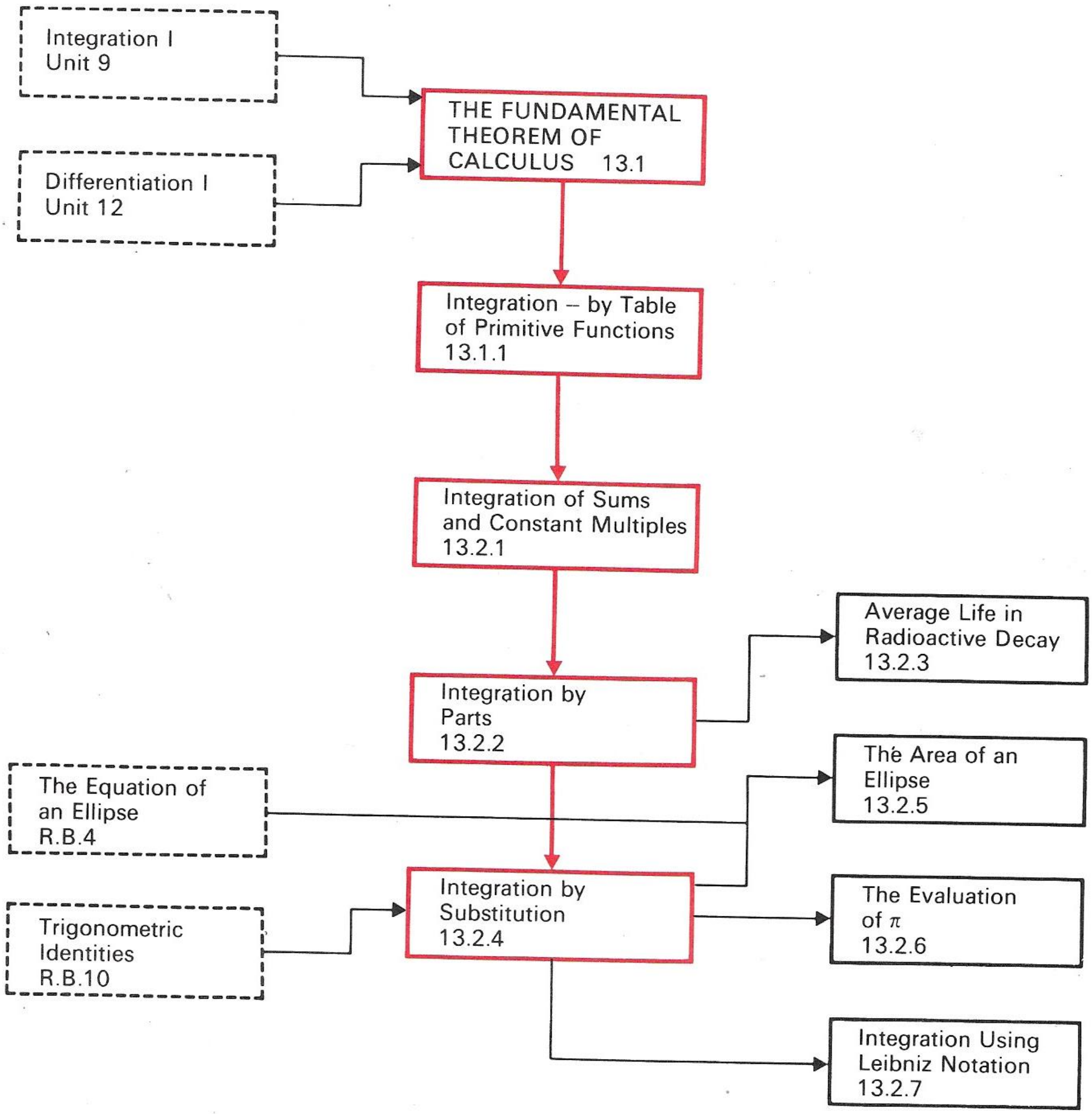
(iv) apply the method of integration by substitution to suitable given integrals;

(v) convert integrals expressed in function notation into Leibniz notation, and vice versa.

N.B.

Before working through this correspondence text, make sure you have read the general introduction to the mathematics course in the Study Guide, as this explains the philosophy underlying the whole course. You should also be familiar with the section which explains how a text is constructed and the meanings attached to the stars and other symbols in the margin, as this will help you to find your way through the text.

Structural Diagram



Glossary

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Terms which are defined in this glossary are printed in CAPITALS.

CONSTANT OF INTEGRATION	All real functions of the form $x \longmapsto F(x) + c, c \in R$, are PRIMITIVES of the real function f (where $DF = f$); c is called a CONSTANT OF INTEGRATION.	5
ELLIPSE	An ELLIPSE is a plane figure which (for an appropriate choice of Cartesian co-ordinates) is the graph of the equation $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$ where x, y are Cartesian co-ordinates and a, b are positive real numbers.	38
IDENTITY	An IDENTITY is a formula such as $f(x) = g(x)$, which relates images under two functions and which holds for *all* elements in their common domain.	41
INDEFINITE INTEGRAL	An INDEFINITE INTEGRAL is an alternative name for a PRIMITIVE FUNCTION.	2
INTEGRATE A GIVEN FUNCTION f	To INTEGRATE A GIVEN FUNCTION f is to find a PRIMITIVE FUNCTION of f.	3
INTEGRATION BY PARTS	INTEGRATION BY PARTS is the evaluation of the integral $\int_a^b f \times Dg$ using the rule: $$\int_a^b f \times Dg = [f \times g]_a^b - \int_a^b g \times Df.$$	21
INTEGRATION BY SUBSTITUTION	See SUBSTITUTION.	
INTEGRATION OPERATOR	The operator, denoted here by I, which maps a real continuous function to its PRIMITIVE FUNCTIONS is called the INTEGRATION OPERATOR.	3
PRIMITIVE FUNCTION	If f is a real function, then a function F with the property that $$\int_a^b f = F(b) - F(a),$$ for all a and b in the domain of f, is called a PRIMITIVE FUNCTION of f.	2
SUBSTITUTION, INTEGRATION BY	Integration by substitution is the evaluation of the integral $\int_a^b (g \circ k) \times Dk$ using the rule: $$\int_a^b (g \circ k) \times Dk = \int_{k(a)}^{k(b)} g.$$	35

Notation

Page

The symbols are presented in the order in which they appear in the text.

$\int_a^b f$	The definite integral of f in $[a, b]$.	2
$R \times R$	The Cartesian product of R with itself.	3
I	The integration operator.	3
$f_{av}[a, b]$	The average value of the function f over the interval $[a, b]$; that is, $$\frac{1}{b-a}\int_a^b f.$$	7
$\lim_{b \to a} f(b)$	The limit of f near the point a.	8
D	The differentiation operator.	8
$\frac{df(x)}{dx}$	The Leibniz notation for the derivative of f at x.	10
$\int_a^b f(x)\,dx$	The Leibniz notation for $\int_a^b f$.	10
$\Rightarrow$	The logic symbol for implication.	10
$[F]_a^b$	$F(b) - F(a)$, that is, $\int_a^b f$, where $DF = f$.	12
exp	The exponential function.	13
$\Leftrightarrow$	The logic symbol for equivalence.	13
$\mathscr{C}$	The domain of the operator I.	15
ln	The logarithm function.	22
$\int_a^b x \cos x$	The definite integral of the function $x \longmapsto x \cos x$ in $[a, b]$.	24
$\int f$	One of the primitive functions of the real continuous function f.	24

Bibliography

Any text book on calculus will contain some kind of treatment of the topics dealt with in this unit. The following books are particularly recommended.

S. K. Stein, *Calculus for the Natural and Social Sciences* (McGraw-Hill 1968).
This is a well-planned book with plenty of diagrams, exercises and applications. The Fundamental Theorem is explained and proved in Chapter 6 (pages 116 to 129). The rules of integration are given in Chapter 7 (pages 130 to 138) and some applications of integration are discussed in Chapter 8.

R. Courant, *Differential and Integral Calculus Vol. I* (Blackie 1966).
This is a very clear exposition of the mathematics and contains plenty of diagrams and exercises. The applications considered on pages 122 to 126 are mainly to physics. The Fundamental Theorem is treated on pages 109 to 119 and the rules of integration are discussed on pages 141 to 144 and 204 to 221.

T. M. Apostol, *Calculus Vol. I* (Blaisdell 1967).
This book is well written and of the type used as a text for a year's work in American universities. A good account of the Fundamental Theorem is given on page 202; integration by substitution and by parts is treated on pages 212 to 220. There are plenty of diagrams and exercises. An excellent discussion of the Leibniz notation is given on page 210.

J. C. Burkill, *A First Course in Mathematical Analysis* (Cambridge University Press 1962).
This is a very elegant and concise treatment of calculus intended for "students who have a working knowledge of calculus and are ready for a more systematic treatment". The Fundamental Theorem is treated on pages 128 to 129, and the techniques of integration and the evaluation of π are discussed on pages 129 to 137. It is an excellent book for students intending to take higher level mathematics courses.

13.0 INTRODUCTION

13.0

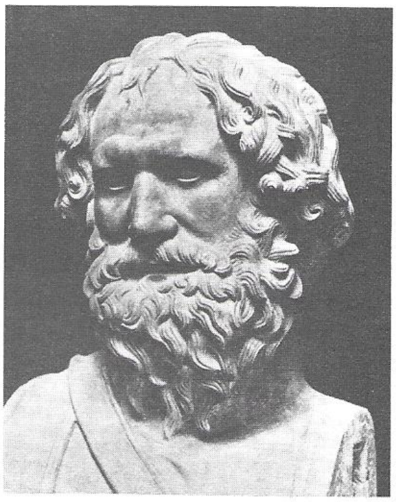
Introduction
* *

This is the second of two units on Integration in the Foundation Course. In *Unit 9, Integration I* we discussed the definite integral, how it could be applied to problems such as the measurement of areas and of volumes of revolution, and how to evaluate definite integrals of polynomial functions directly from the definition of a definite integral. This method of evaluating integrals is based on ideas developed by Archimedes about 2200 years ago, and for about 1900 years after his death it was the only method available for such problems. About 300 years ago, however, Newton and Leibniz independently discovered a much more powerful method, which greatly simplifies the calculations and, in consequence, greatly enlarges the class of functions that can be integrated. The present unit is devoted to the study of this second method of evaluating integrals.

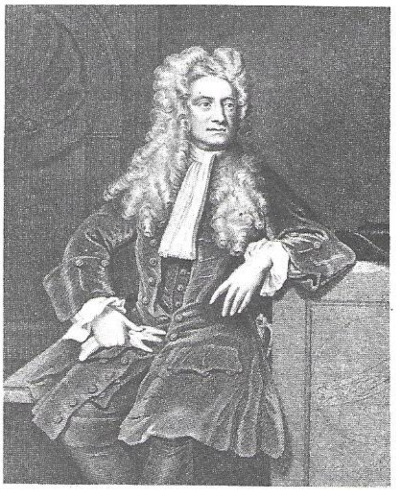
The basis of the method is a theorem known as the Fundamental Theorem of Calculus. This theorem establishes a connection between two subjects which we have hitherto treated in isolation: integration and differentiation.

The reason why this theorem took 1900 years to discover is that it required several concepts that were, in Newton's time, unfamiliar and difficult even for the greatest mathematicians. Today, however, the situation is quite different; we are familiar with the concept of a limit, and know how it can be used to define the terms *derivative* and *definite integral*. With these milestones behind us, we should not find the Fundamental Theorem too difficult.

In the first part of this text we shall discuss the Fundamental Theorem. In the second part, we shall introduce some of the powerful techniques of integration that were made possible by the discovery of the Fundamental Theorem. In this latter part we shall also consider some applications of these methods to problems outside the calculus itself. For example, the method makes it possible to write down an exact formula for the area of a circle, and hence to express π as a definite integral. From this formula one can (given sufficient time and patience) calculate π as accurately as one pleases.



Archimedes 287–212 B.C.
(Mansell Collection)



Isaac Newton 1642–1727
(Mansell Collection)

Gottfried Wilhelm Leibniz
1646–1716
(Mansell Collection)

13.1 THE FUNDAMENTAL THEOREM OF CALCULUS 13.1

13.1.1 Primitive Functions 13.1.1

Main Text

At first sight there seems to be no connection at all between differentiation and integration, or between tangents to curves and the areas under the curves. The first step in seeing that there actually *is* a connection is to look more closely at the structure of the formulas for integrals obtained earlier in the course.

In *Unit 9, Integration I* we saw that one way of looking at the definite integral is as an area, and by approximating areas by sums of rectangles we were able to find exact formulas for a few definite integrals, for example:

$$\int_a^b x \longmapsto 1 = b - a,$$

$$\int_a^b x \longmapsto x = \frac{b^2}{2} - \frac{a^2}{2},$$

$$\int_a^b x \longmapsto x^2 = \frac{b^3}{3} - \frac{a^3}{3},$$

where a and b are real numbers, and the various functions are all real functions (that is, with domain and codomain R or a subset of R; the domain must, of course, include the interval $[a, b]$). Although the expressions on the right-hand sides of the three equations are all different, they have a common feature: each of them is the difference of two terms, one depending on b and the other depending *in the same way* on a. Let us use the letter F to denote the real function which specifies the way in which the first term on the right depends on b (for example, $F: b \longmapsto \frac{b^2}{2}$ in the second formula); then the first term on the right is $F(b)$, and the second is $F(a)$, and each of the formulas can be written in the form:

$$\int_a^b f = F(b) - F(a)$$

Equation (1)

with suitable functions f and F. We shall call the function F a primitive function* of the function f; in our third example, $b \longmapsto \frac{b^3}{3}$ is a primitive function of $x \longmapsto x^2$. In general, given any continuous† real function f, we define a primitive function of f to be any F such that Equation (1) holds for all a and b in the domain of f.

Definition 1

Notice that we say "*a* primitive function", not "*the* primitive function". This is because primitive functions are not unique: for each f there are *many* primitive functions F. For instance, instead of

$$F: b \longmapsto \frac{b^2}{2}$$

in the second example above, we could choose

$$F_1: b \longmapsto \frac{b^2}{2} + 3$$

and still have

$$\int_a^b x \longmapsto x^2 = F_1(b) - F_1(a),$$

and so F_1 is also a primitive function of f.

* The term indefinite integral is common. Our terminology is meant to emphasize that F is a *function*, not a number like a definite integral.

† For the definition of continuity, see *Unit 7, Sequences and Limits I*.

Exercise 1

Exercise 1
(2 minutes)

If f is a continuous real function with a primitive function F, use Equation (1) to show that

$$\int_a^b f = -\int_b^a f$$

for all a and b in the domain of f. ■

Exercise 2

Exercise 2
(2 minutes)

Complete the following table:

f	$F(b)$	$F(a)$	F, a primitive function of f
$x \longmapsto 1 \quad (x \in R)$	b	a	$x \mapsto x$.
$x \longmapsto x \quad (x \in R)$	$b^2/2$	$a^2/2$.	$x \mapsto \frac{x^2}{2}$
$x \longmapsto x^2 \quad (x \in R)$	$b^3/3$	$a^3/3$	$x \mapsto \frac{x^3}{3}$

■

Main Text

For a given continuous real function f with domain R, the definite integral $\int_a^b f$ is determined by the values of both a and b; evaluating it is therefore tantamount to calculating the image of (a, b) under the following function of two variables:

$$(a, b) \longmapsto \int_a^b f \qquad ((a, b) \in R \times R).$$

If we regard either a or b as fixed, then we can consider the definite integral as defining a function of one variable, with domain R rather than $R \times R$; for example,

$$b \longmapsto \int_a^b f \qquad (b \in R).$$

Functions with domain R are usually easier to deal with than those with domain $R \times R$, but it is not clear at the moment just how this new function is going to help. This is where the Fundamental Theorem of Calculus comes in: it gives us a general method for finding a primitive function of f *without* first evaluating the integral by summing rectangles.

The table in Exercise 2 is reminiscent of the table of derivatives in *Unit 12, Differentiation I*. The first and last columns constitute a list of ordered pairs of functions, and may therefore be held to define a mapping whose domain and codomain are sets of functions — that is, an *operator*. There is no need to restrict the domain of this operator to the three functions listed in Exercise 2; rather, we may expect to be able to use for the domain some much more general set of functions f, such that $\int_a^b f$ exists for all a and b in the domain of f. (A suitable set is the set of all real continuous functions. See the end of section 13.1.4.) This operator may be called the integration operator, and represented symbolically as follows:

Definition 2

$$I : f \longmapsto (\text{the set of all primitive functions of } f).$$

The process of finding a primitive function is called integration, and in applying I to f one is said to integrate the function f.

Definition 3

Solution 1

Solution 1

If F is a primitive function of f, then

$$\int_a^b f = F(b) - F(a)$$

for all a and b in the domain of f.

It follows that

$$\int_b^a f = F(a) - F(b)$$

$$= -(F(b) - F(a)),$$

that is,

$$\int_b^a f = -\int_a^b f.$$

■

Solution 2

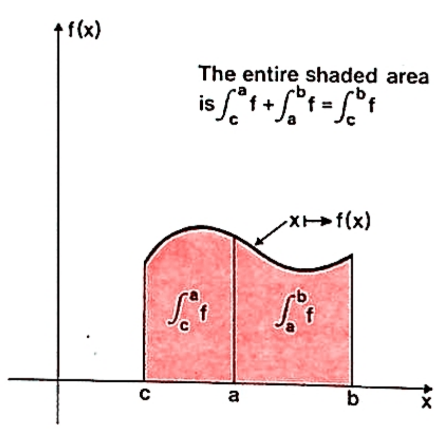
Solution 2

f	$F(b)$	$F(a)$	F, a primitive function of f
$x \longmapsto 1 \quad (x \in R)$	b	a	$x \longmapsto x \quad (x \in R)$
$x \longmapsto x \quad (x \in R)$	$\dfrac{b^2}{2}$	$\dfrac{a^2}{2}$	$x \longmapsto \frac{1}{2}x^2 \quad (x \in R)$
$x \longmapsto x^2 \quad (x \in R)$	$\dfrac{b^3}{3}$	$\dfrac{a^3}{3}$	$x \longmapsto \frac{1}{3}x^3 \quad (x \in R)$

■

Exercise 3

Exercise 3
(3 minutes)



Use the result $\int_c^a f + \int_a^b f = \int_c^b f$ (given in *Unit 9, Integration I*; see also the above diagram) to show that, if f is a real continuous function with domain R, then for any real number c the function F given by

$$F: x \longmapsto \int_c^x f \qquad (x \in R)$$

is a primitive function of f.

■

Exercise 4

Exercise 4
(3 minutes)

If f is a real continuous function with domain R, and F is a primitive function of f, are there any numbers c (other than zero) for which the function F_c defined by

$$F_c : x \longmapsto F(x) + c \qquad (x \in R)$$

is also a primitive function of f? ■

Main Text

The result of this last exercise is an important one. If F is any primitive function of some given function f, then any function of the form

$$x \longmapsto F(x) + c \qquad (x \in \text{domain of } F)$$

where c is any real number, is also a primitive function of f. Another way of saying the same thing is that the operator I is not a function: under this operator the image $I(f)$ of a given element f in the domain of I is not a *unique* element of the codomain of I, but a *set* of such elements. The real number c is called a constant of integration, and each different value for c gives a different primitive function of f.

Definition 4

Exercise 5

Exercise 5
(2 minutes)

Find a primitive function F of the function

$$x \longmapsto x \qquad (x \in R)$$

with the property $F(0) = 1$. ■

13.1.2 The Fundamental Theorem of Calculus: Part 1

13.1.2

Main Text

If the only property of primitive functions F of a function f were that they satisfied the definition given in the preceding section; that is,

$$\int_a^b f = F(b) - F(a) \qquad ((a, b) \in R \times R),$$

Equation (1)

they would provide little more than a useful alternative notation for definite integrals. They would not help us with the job of actually calculating the definite integrals, because our only way of finding a primitive function of f would be to find the definite integral first, and then use Equation (1), or the formula in Exercise 13.1.1.3, to find F. The property hich makes the primitive function concept really useful is that there is another way of finding primitive functions, which does not require us find the corresponding definite integral first. This method is provided by the Fundamental Theorem of Calculus.

The basic idea of the Fundamental Theorem is to creep up on the integration mapping from behind, as it were, by identifying its reverse mapping. view of the property we used to define I,

$$I : f \longrightarrow F,$$

this is equivalent to finding a rule giving f in terms of one of its primitives. other words, we shall regard F (rather than f) as the given function Equation (1), and try to determine from it the function f. This will enable us to identify the mapping

$$F \longmapsto f;$$

we can then find primitives of f by reversing this new mapping instead by evaluating definite integrals directly.

(*continued on page 6*)

Solution 13.1.1.3

Solution 13.1.1.3

The result $\int_c^a f + \int_a^b f = \int_c^b f$ gives

$$\int_a^b f = \int_c^b f - \int_c^a f$$

$$= F(b) - F(a) \qquad \text{by definition of } F,$$

and hence F is a primitive function of f. Notice that, since c is arbitrary, this method enables us to define as many different primitive functions of f as we please. ■

Solution 13.1.1.4

Solution 13.1.1.4

Yes: *any* real number c gives a primitive function of f. To test whether F_c is a primitive function of f we must test whether

$$\int_a^b f = F_c(b) - F_c(a) \qquad ((a, b) \in R \times R).$$

Since F is a primitive function of f, we have:

$$\int_a^b f = F(b) - F(a) \qquad ((a, b) \in R \times R),$$

and therefore

$$\int_a^b f = (F(b) + c) - (F(a) + c) \qquad ((a, b) \in R \times R);$$

that is,

$$\int_a^b f = F_c(b) - F_c(a) \qquad ((a, b) \in R \times R),$$

so F_c is a primitive function of f. ■

Solution 13.1.1.5

Solution 13.1.1.5

A primitive function of $x \longmapsto x$ is $x \longmapsto \frac{1}{2}x^2$, so that a more general primitive function of $x \longmapsto x$ is $x \longmapsto \frac{1}{2}x^2 + c$, where c is any real number. Denoting this function by F, we have

$$F(x) = \tfrac{1}{2}x^2 + c, \text{ and hence } F(0) = c.$$

The exercise requires $F(0) = 1$, so that $c = 1$, and therefore the required primitive function is

$$x \longmapsto \tfrac{1}{2}x^2 + 1 \qquad (x \in R).$$ ■

(*continued from page 5*)

We assume as usual that f and F are real functions, and we shall also assume, in order to be able to state theorems that can be proved rigorously (even though we do not prove them rigorously here), that f is continuous everywhere in its domain. Since there is no immediately obvious way to go about finding f from F using Equation (1), this is a good place to apply one of Polya's problem-solving maxims: "Do you know a related problem?" (see *Polya**, pages xvi and 98). We need to be able to get the value of $f(x)$ from an integral of f. We cannot do this yet, but in section 9.3.2 of *Unit 9, Integration I* we did see how to get *average values* from

* G. Polya, *How to Solve It*, Open University ed. (Doubleday Anchor Books 1970). This book is the set book for the Mathematics Foundation Course; it is referred to in the text as *Polya*.

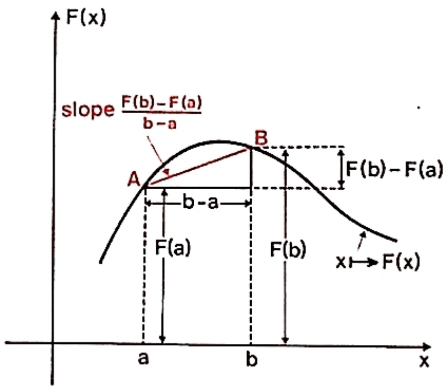
definite integrals. The formula was

$$f_{av}[a, b] = \frac{1}{b - a}\int_a^b f,$$

where $f_{av}[a, b]$ denotes the average value of the function f over the interval $[a, b]$. Equation (1) now gives us this average value in terms of F:

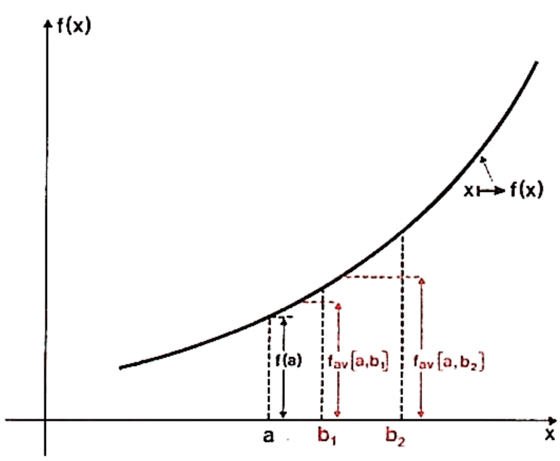
$$f_{av}[a, b] = \frac{F(b) - F(a)}{b - a}.$$ Equation (2)

The expression on the right can be interpreted graphically: it is the slope of a chord of the graph of F, as illustrated in the following figure:



Finding the average value of f(x) from the graph of F

We now need a way to get from the *average* value of f over $[a, b]$ to the value of f *at some specific point* in its domain. We faced a similar problem earlier in the course (*Unit 7, Sequences and Limits I*; *Unit 12, Differentiation I*) when we wanted to obtain *instantaneous* velocities from *average* velocities, and the method is just the same here. To refresh your memory, we shall go quickly through the argument again. Since f is continuous, we can argue that, if a and b are very close together, then $f(x)$ is very nearly constant over the interval $[a, b]$, so that its value at any specific point in the interval, say a, is closely approximated by the average value over $[a, b]$, as illustrated in the figure below:



How the average value of f(x) varies as the interval width is reduced

By making b close enough to a we expect to make the error in this approximation as small as we please, and by a suitable limiting procedure we expect to obtain $f(a)$ exactly. To formulate this idea more precisely, we expect to find that

$$\lim_{b \to a} (f_{av}[a, b]) = f(a).$$ Equation (3)

that is

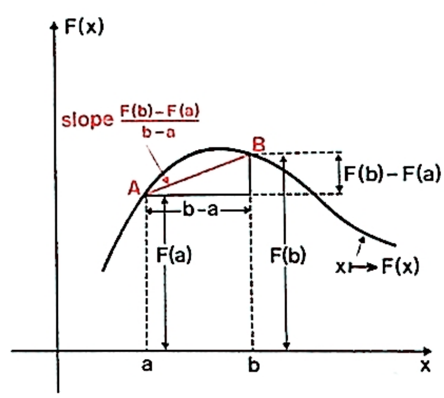
$$\lim_{b \to a} \left(\frac{1}{b-a} \int_a^b f \right) = f(a).$$

It can be shown that this equation does indeed hold (and that the limit on the left exists), when the function f is continuous at a. If you are interested in such a proof, see Apostol, *Calculus Vol. I*, p. 202 or Burkill, *Mathematical Analysis*, p. 129. (These books are described in the Bibliography.)

Substituting from Equation (2) for the left-hand side of Equation (3), we find

$$\lim_{b \to a} \frac{F(b) - F(a)}{b - a} = f(a).$$

In terms of the graph of F, the limit on the left is the limiting slope of the chord AB when B is very close to A. We know already from *Unit 12, Differentiation I* that this is the slope of the tangent at A, and is given by the derivative of the function F at a, that is, by $DF(a)$, where D is the differentiation operator defined in *Unit 12, Differentiation I.*



Thus we have found that

$$DF(a) = f(a).$$

Since this equation holds for any real a in the domain of f, it follows that the functions DF and f are identical. This is the first part of the Fundamental Theorem of Calculus:

Theorem

> If f is a real continuous function and if F is a primitive function of f, then $DF = f$.

In other words, the differentiation which takes us from F to f *undoes* the integration which took us from f to F.

Example 1

Example 1

We have already seen that a primitive function of $(x \longmapsto x)$ is $(x \longmapsto \frac{1}{2}x^2)$. How does this fit in with the theorem?

In the context of the theorem, $(x \longmapsto x)$ is f and $(x \longmapsto \frac{1}{2}x^2)$ is F. According to the theorem, $DF = f$, and indeed we see here that

$$D(x \longmapsto \tfrac{1}{2}x^2) = (x \longmapsto x).$$ ■

Exercise 1

Exercise 1
(5 minutes)

Find the derivative at x of each of the functions

$$x \longmapsto \int_a^x f \quad \text{and} \quad x \longmapsto \int_x^b f,$$

where f is a real function and a and b belong to the domain of f.
(HINT: Reconsider Exercise 13.1.1.3.)

If you are familiar with the Leibniz notation for derivatives, write your results in this notation too.

These results give an alternative, very convenient formulation of the first part of the Fundamental Theorem. ■

Exercise 2

Exercise 2
(3 minutes)

Use the Fundamental Theorem to check whether the following statements are true or false:

(i) $\int_a^b x \longmapsto \sin x = \cos b - \cos a$ TRUE/FALSE

(ii) $\int_a^b x \longmapsto \cos x = \sin b - \sin a.$ TRUE/FALSE

■

Main Text

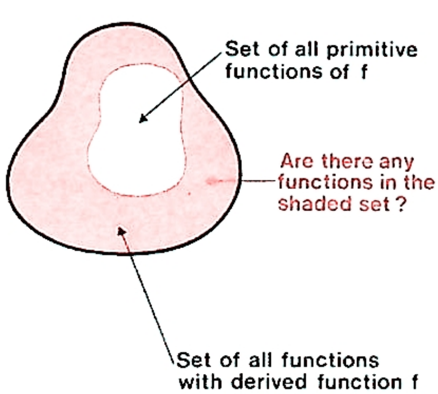
The first part of the Fundamental Theorem of Calculus does not completely solve the problem of finding primitive functions, but it takes us a long way towards the solution. It does not show us how to *find* a primitive function, F, of a given continuous function f. It *does* tell us that each primitive function F can be differentiated, and has derived function f. We can use the theorem to find f when F is known, or to check the calculation by which a primitive function has been found.

13.1.3 The Fundamental Theorem of Calculus: Part 2

13.1.3

Discussion

How can we use the result of the preceding section to evaluate integrals? To evaluate a definite integral involving a given function f, it is sufficient to know a primitive function of f; the result in question helps us to recognize a possible primitive function, by telling us that every primitive function of a given continuous function f has the property that its derived function is f. Accordingly, if we look among the functions which have derived function f, we shall find all the primitives of f, but perhaps some other functions as well. Thus the result narrows the field in which to search for primitive functions of f, but it does not tell us how to be sure of finding them, or even how to be sure whether a supposed primitive function of f really is one or not.



(*continued on page 11*)

Solution 13.1.2.1

Solution 13.1.2.1

Writing

$$F_1 \text{ for the function } x \longmapsto \int_a^x f$$

and

$$F_2 \text{ for } x \longmapsto \int_x^b f,$$

we see (by the result of Exercise 13.1.1.3) that F_1 is a primitive function of f, so that the Fundamental Theorem gives

$$DF_1 = f,$$

that is, the derivative of $x \longmapsto \int_a^x f$ at x is $f(x)$.

For F_2, we can use the result:

$$\int_x^b f = -\int_b^x f$$

obtained in Exercise 13.1.1.1; this gives

$$F_2 : x \longmapsto -\int_b^x f;$$

that is,

$$-F_2 : x \longmapsto \int_b^x f.$$

Using the result of Exercise 13.1.1.3 again, we see that $-F_2$ is a primitive function of f, and so

$$D(-F_2) = f$$

By the First Rule of Differentiation (see *Unit 12*, section 12.2.2),

$$D(-1 \times F_2) = -1 \times DF_2,$$

and therefore

$$DF_2 = -f;$$

that is, the derivative of $x \longmapsto \int_x^b f$ at x is $-f(x)$. In Leibniz notation these results may be written:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x) \quad \text{and} \quad \frac{d}{dx}\int_x^b f(t)\,dt = -f(x),$$

where t is a dummy variable. ■

Solution 13.1.2.2

Solution 13.1.2.2

(i) FALSE. The statement asserts that the cosine function is a primitive function of the sine function. Differentiating the primitive, cos, should restore the original function, sin, but in fact we have $D\cos = -\sin$, so the assertion given is false.

(ii) The statement asserts that sin is a primitive of cos; if this is so, then we should have $D\sin = \cos$, which is true; so there is no evidence against the assertion and we mark it TRUE. Notice the caution implied by our choice of words. We have argued in the previous text that*

$$(F \text{ is a primitive function of } f) \Rightarrow (DF = f),$$

* $\Rightarrow$ is the logic symbol for implication, introduced in *Unit 11*, *Logic I*.

but we have not shown that

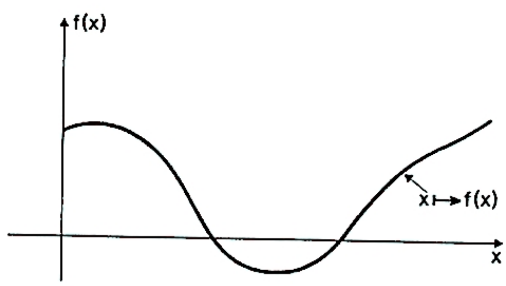
$$(DF = f) \Rightarrow (F \text{ is a primitive function of } f)$$

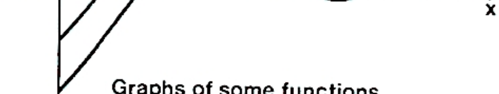
which is the result we require here. ■

(*continued from page 9*)

In this section we shall demonstrate a further result which removes any doubt, by showing that under suitable conditions the "other functions" referred to (p. 9) do not exist: every function with derived function f is in fact a primitive function of f. In other words we shall show that the shaded region in our Venn diagram represents an empty set.

The principal step is to characterize the set in which the primitive functions of f are to be found: the set of all functions with derived function f. In terms of graphs, this set is the set of all functions whose graphs have slope $f(\alpha)$ at each point with x-co-ordinate α. The following diagram shows, at the top, the graph of a continuous function f and, at the bottom, graphs of a few functions with derived function f.





Graphs of some functions with derived function f

This diagram indicates that the functions with derived function f have graphs that are congruent curves. That is, any one of the curves can be superimposed on any other by shifting it in a direction parallel to the y-axis: such a shift alters neither the x-co-ordinate of any point on the curve nor the slope, $f(x)$, at that point.

Such a shift in the graph is equivalent to adding a constant function to the original function; that is, replacing a function such as $x \longmapsto F(x)$ by $x \longmapsto F(x) + c$ where c is a real constant giving the amount of the shift. This demonstrates that all the functions with derived function f differ by constant functions. Like most arguments based on diagrams, this is a demonstration, not a proof. For a proof see Apostol, *Calculus Vol. I*, p. 187 or Burkill, *Mathematical Analysis*, p. 75.

So if F and F_1 both have derived function f, they differ only by a constant function. But we have seen in Exercise 13.1.1.2 that if F is a primitive of f, then any function that differs from F only by a constant function is also a primitive. Thus every function that has derived function f is a primitive of f, and so the shaded region of the Venn diagram on page 9 represents an empty set.

The result that we have just demonstrated is the second part of the Fundamental Theorem of Calculus. It tells us that, to find some primitive function of a given continuous function f, it is sufficient to find any function whose derivative is f. A concise statement of the result can be obtained by denoting one of the functions with derived function f by F, so that $f = DF$; then the result tells us that F is a primitive of DF, or more precisely that

If F is a real function whose domain includes the interval $[a, b]$, and if DF is continuous in $[a, b]$, then

$$\int_a^b DF = F(b) - F(a).$$

Theorem

Because expressions like $F(b) - F(a)$ occur frequently, we abbreviate by writing

$$[F]_a^b = F(b) - F(a)$$

Notation 1

so that, for example

$$[x \longmapsto x^2]_2^3 = 3^2 - 2^2 = 5.$$

To illustrate how this second part of the Fundamental Theorem of Calculus is used to evaluate integrals, let us apply it to

$$\int_1^2 x \longmapsto x^3.$$

We look for a function F such that

$$DF = x \longmapsto x^3.$$

We showed in *Unit 12* that differentiation always reduces the degree of a polynomial function by one, so we are led to consider $D(x \longmapsto x^4)$, which is $x \longmapsto 4x^3$. Apart from the factor 4, this is just what we want, and so a suitable function F is $x \longmapsto \frac{1}{4}x^4$. Hence, we have

$$\begin{aligned}\int_1^2 x \longmapsto x^3 &= \int_1^2 D(x \longmapsto \tfrac{1}{4}x^4)\\ &= [x \longmapsto \tfrac{1}{4}x^4]_1^2\\ &= \tfrac{1}{4}(2^4) - \tfrac{1}{4}(1^4)\\ &= \tfrac{15}{4}.\end{aligned}$$

The function $x \longmapsto x^3$ is continuous, so our application of the theorem is justified.

Exercise 1

Exercise 1
(3 minutes)

Using the Fundamental Theorem of Calculus, evaluate

$$\int_a^b x \longmapsto x.$$

Look back at the solution to Exercise 9.2.1.3 of *Unit 9, Integration I* and compare the amount of calculation needed there (to evaluate the same integral without using the Fundamental Theorem of Calculus) with the amount needed here.

■

Exercise 2

Exercise 2
(5 minutes)

Use the Fundamental Theorem of Calculus and the table of standard derived functions, given in *Unit 12*, section 12.4.1, to evaluate

(i) $\int_0^r \exp$ (ii) $\int_0^{\pi} \cos$ (iii) $\int_0^{\pi/2} \sin$.

Could you have evaluated the second integral in a simpler way?
(HINT: Draw the graph of the cosine function and interpret the integral in terms of the graph; use the symmetry of the curve.)

13.1.4 The Complete Fundamental Theorem

13.1.4

Discussion

The first and second parts of the Fundamental Theorem of Calculus tell us, respectively, that differentiation "undoes" integration and that integration "undoes" differentiation. In this section we shall combine the two parts of the theorem into a single statement which expresses the symmetrical relation between integration and differentiation. By expressing the two parts of the Fundamental Theorem in terms of the differentiation and integration operators, we shall show that the theorem as a whole states that these two mappings are reverses of each other.

The essential properties of the differentiation and integration operators are that the differentiation operator maps functions to their derived functions, and the integration operator maps functions to their primitive functions. In symbols, these properties can be written:

$$D: F \longmapsto \text{the derived function of } F$$

$$I: f \longmapsto \text{the set of all primitive functions of } f.$$

The domains of the two operators are both sets of real functions; we shall specify these sets more precisely later.

The first part of the Fundamental Theorem of Calculus, given in section 13.1.2 (page 8) states (for a suitable class of functions f): if F is a primitive function of f, then f is the derived function of F.

In terms of the "implies" symbol, we have:

$$(I: f \longrightarrow F) \Rightarrow (D: F \longmapsto f).$$

The second part of the Fundamental Theorem, given in section 13.1.3 (page 12) states (for a suitable class of functions F): if f is the derived function of F, then F is a primitive function of f. We therefore have:

$$(D: F \longmapsto f) \Rightarrow (I: f \longrightarrow F).$$

Taking these two statements together, and using the logic symbol of equivalence we introduced in *Unit 11, Logic I*, we have:

$$(D: F \longmapsto f) \Leftrightarrow (I: f \longrightarrow F).$$

This result provides us with a unified statement of the complete Fundamental Theorem of Calculus. Stated in words, it is (for continuous f with suitable domain and codomain):

Fundamental Theorem

> f is the derived function of F if and only if F is a primitive function of f.

Even more concisely, we can say that

> the mappings D and I are reverse mappings

(see *Unit 1, Functions*).

(*continued on page 15*)

Solution 13.1.3.1

Solution 13.1.3.1

$$\int_a^b x \longmapsto x = \int_a^b D(x \longmapsto \tfrac{1}{2}x^2) = [x \longmapsto \tfrac{1}{2}x^2]_a^b = \tfrac{1}{2}b^2 - \tfrac{1}{2}a^2.$$

The solution given in *Unit 9, Integration I* is considerably longer than this one! ■

Solution 13.1.3.2

Solution 13.1.3.2

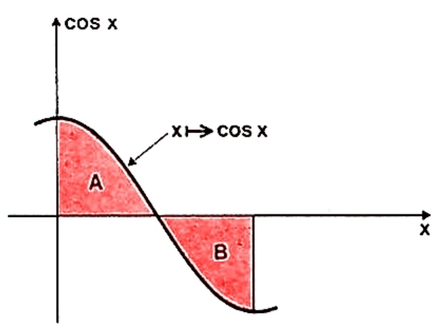
(i) $\int_0^t \exp = \int_0^t D\exp = [\exp]_0^t = \exp(t) - \exp(0) = e^t - 1$

(ii) $\int_0^\pi \cos = \int_0^\pi D\sin = [\sin]_0^\pi = \sin\pi - \sin 0 = 0 - 0 = 0$

(iii) $\int_0^{\pi/2} \sin = \int_0^{\pi/2} D(-\cos) = [-\cos]_0^{\pi/2} = -\cos\frac{\pi}{2} + \cos 0$

$= -0 + 1 = 1.$

An alternative method for the second case is:



$$\int_0^\pi \cos = \int_0^{\pi/2} \cos + \int_{\pi/2}^\pi \cos$$

$$= \text{area } A - \text{area } B$$

$$= 0, \quad \text{by symmetry.}$$

The area B contributes negatively to the integral because the curve is below the x-axis. The total *area* is

$$\int_0^{\pi/2} \cos - \int_{\pi/2}^\pi \cos = [\sin]_0^{\pi/2} - [\sin]_{\pi/2}^\pi = 2. \quad ■$$

Exercise 1

Indicate the property of each of the mappings by ticking the appropriate box:

Exercise 1 (2 minutes)

	one–one	one–many	many–one	many–many
D				
I				
$D \circ I$				
$I \circ D$				

■

Exercise 2

Exercise 2 (3 minutes)

If F is a function in the domain of the mapping D, what is its image under the mapping $I \circ D$? ■

Discussion

To complete the discussion, we shall specify domains for the mappings D and I. If you have not studied calculus before, omit the rest of this section, read section 13.1.5, and then proceed to the beginning of section 13.2.0.

We are primarily interested in integration here, and so we begin with I. We shall require any function f in the domain of I to satisfy the following three conditions:

(i) the codomain of f must be R or a subset of R;
(ii) the domain of f must be R or an interval* in R;
(iii) the function f must be continuous everywhere in its domain.

The first condition is necessary for an integral of the function f, as defined in *Unit 9, Integration I*, to have a meaning. The second condition is not strictly essential, but it is very convenient because it ensures that, if a and b are two points in the domain of f, then $f(x)$ is defined everywhere in the interval $[a, b]$, so that the integral $\int_a^b f$ is meaningful. The third condition is imposed because both parts of the Fundamental Theorem require the function that is integrated to be continuous. We take the domain of I to be the set of all functions satisfying the conditions (i), (ii) and (iii); this set will be denoted by the script letter $\mathscr{C}$.

Notation 1

Since the mappings D and I are reverses of each other, the domain of D is now determined: it must be the image of the domain of I, namely $I(\mathscr{C})$, which is the set of all primitive functions of functions belonging to $\mathscr{C}$. By the Fundamental Theorem, this set is identical with the set of all functions whose derived functions satisfy the three conditions (i), (ii) and (iii) above. The corresponding conditions on a function F in this set are:

(i′) the codomain of F must be R or a subset of R;
(ii′) the domain of F must be R or an interval* of R, and the derivative of F must exist everywhere in the domain of F;
(iii′) the derived function of F must be continuous everywhere in its domain.

The domain of D defined above is not quite the same as the domain for the differentiation operator given in *Unit 12, Differentiation I*, but we have used the same symbol D for both differentiation operators, because their similarities are more important than their differences.

* Here we use the term "interval" (in a wider sense than before) to mean any subset of R containing more than one number, and also containing all the numbers between any two of its members; for example R^+, R^-, $[a, b]$, $[a, b[$, etc.

(*continued on page 17*)

Solution 1

Solution 1

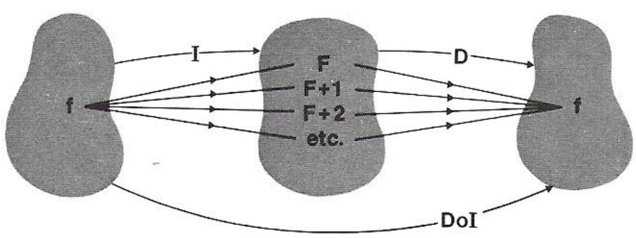
	one–one	one–many	many–one	many–many
D			√	
I		√		
$D \circ I$	√			
$I \circ D$				√

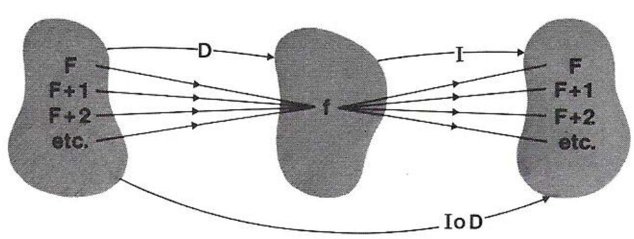
The mapping D is many–one: the derived function of a given function is unique, but many different functions (differing by constant functions) have the same derived function.

The mapping I, being the reverse of D, is one–many.

The image of a continuous function f under the mapping I, which we denote by $I(f)$, is the set of all primitives of f. By the first part of the Fundamental Theorem, every one of these primitives has derived function f, and so the composite mapping $D \circ I$ is one–one.

Finally, the mapping $I \circ D$, being the composite of a many–one mapping followed by a one–many mapping, is many–many; this mapping is studied in more detail in the next exercise.





■

Solution 2

Solution 2

The image of F under the mapping $I \circ D$ is the image of DF under the mapping I; that is, it is the set of all primitive functions of DF. According to the Fundamental Theorem, this set is identical with the set of all functions with derived function DF. One function in this set is F itself, and the others are all the functions of the form $x \longmapsto F(x) + c$, where c is any real number. We conclude that the image of F under the mapping $I \circ D$ is the set of all functions of the form $x \longmapsto F(x) + c$, where c is any real number. ■

(continued from page 15)

Exercise 3

Which of the following are valid applications of the Fundamental Theorem of Calculus? If an application is not valid, explain why the theorem is not applicable.

Exercise 3
(This exercise is quite hard, but do not spend more than 10 minutes on it.)

(i) $\displaystyle\int_{-1}^{1} f_1 = [F_1]_{-1}^{1}$ where $F_1 : x \longmapsto \begin{cases} \tfrac{1}{2}x^2 \text{ if } x > 0 \\ 0 \text{ if } x = 0 \\ -\tfrac{1}{2}x^2 \text{ if } x < 0 \end{cases}$ $\quad (x \in R)$

and $f_1 : x \longmapsto |x| \quad (x \in R)$.

(ii) $\displaystyle\int_{-1}^{1} f_2 = [f_1]_{-1}^{1}$ where $f_2 : x \longmapsto \begin{cases} 1 \text{ if } x \geqslant 0 \\ -1 \text{ if } x < 0 \end{cases}$ $\quad (x \in R)$

and f_1 is given in (i).

(iii) $\displaystyle\int_{-1}^{1} f_3 = [f_2]_{-1}^{1}$ where $f_3 : x \longmapsto 0 \quad (x \in R \text{ and } x \neq 0)$

and f_2 is given in (ii).

(iv) $\displaystyle\int_{-1}^{1} f_4 = [f_2]_{-1}^{1}$ where $f_4 : x \longmapsto 0 \quad (x \in R)$

and f_2 is given in (ii). ■

13.1.5 Summary

13.1.5

Summary

If f is any real continuous function, then a function F defined by

$$F(x) = \int_a^x f + c \qquad (x \in \text{domain of } f),$$

where a and c are real numbers and a belongs to the domain of f, is called primitive function of f. Primitive functions can be used to evaluate integrals by means of the formula:

$$\int_a^b f = F(b) - F(a) = [F]_a^b,$$

where F is any primitive function of f.

Any two primitive functions of the same f differ by a constant.

The Fundamental Theorem of Calculus can be summarized by the statement (valid if f is a continuous function with codomain R and domain R or an interval of R) that F is a primitive function of f if and only f is the derived function of F; this statement is equivalent to the pair of formulas

$$D\left(x \longmapsto \int_a^x f\right) = f$$

$$\int_a^b DF = F(b) - F(a).$$

The second of these formulas is particularly useful because it enables us to evaluate any integral if we can express the integrand (that is, the function to be integrated) as the derived function of another function.

Solution 13.1.4.3

(i) Valid. f_1 is continuous everywhere in its domain, and F_1 is continuous and differentiable everywhere in its domain.

When $x < 0$, $|x| = -x$, and we have $DF_1(x) = -x$.

When $x \geqslant 0$, $|x| = x$, and we have $DF_1(x) = x$.

It follows that $DF_1 = f_1$.

(ii) Not valid (although the equation given is in fact true), because f_2 is not continuous (its graph has a gap at 0). To derive the given equation from the Fundamental Theorem, the integral must first be split into two parts: $\int_{-1}^{1} f_2 = \int_{-1}^{0} f_2 + \int_{0}^{1} f_2$, and the Fundamental Theorem applied to each part separately.

(iii) Not valid, because there is a gap in the domain of f_3. In fact we have $\int_{-1}^{1} f_3 = 0$, but $[f_2]_{-1}^{1} = 2$.

(iv) Not valid, because f_4 is not the derived function of f_2 (f_2 has no derivative at 0). Again, $\int_{-1}^{1} f_4 = 0$, but $[f_2]_{-1}^{1} = 2$. ■

13.2 THE RULES OF INTEGRATION

13.2

13.2.0 Introduction

13.2.0

Introduction

The discovery of the Fundamental Theorem of Calculus greatly increased the power of analytical (as opposed to numerical) methods in mathematics, by making it possible to evaluate many new integrals that arose in pure mathematics and its applications, particularly in dynamics. One example of a problem leading to integrals requiring these methods is the problem of showing that the gravitational force of attraction between two spherical bodies, such as the earth and the moon, is the same as it would be if the masses of the two bodies were concentrated at their centres; the solution of this problem was an important step in Newton's great achievement of explaining the motions of the heavenly bodies in terms of gravitation. Another such problem is that of calculating π accurately: a very convenient method is to find an integral that is exactly equal to π, and then to evaluate this integral approximately by some method such as Simpson's rule. In both cases the essential point is that techniques based on the Fundamental Theorem of Calculus make it possible to evaluate the integral in question exactly (even though the value of the integral may turn out to be an irrational number such as π).

The purpose of this section is to introduce you to these powerful techniques of integration. They are all based on the following formula which we obtained in section 13.1.3:

$$\int_a^b DF = F(b) - F(a),$$

or on the equivalent statement about primitive functions:

F is a primitive function of DF.

Thus the technical problem in evaluating an integral is to find a function F, given the derived function DF. If the given function DF is sufficiently simple, it may be possible to find it in a table of standard integrals such as the one at the end of this text. Such a table is a list of the functions DF that are derived functions of simple functions F, and is therefore little more than a table of standard derived functions, such as the one given in section 12.4.1 of *Unit 12, Differentiation I*, with the left and right-hand columns interchanged. This interchange of columns mirrors the fact that D and I are reverse mappings.

In most cases, however, it is not possible to find the given function DF in a table of standard integrals. The technique here is to try to express the given integral in terms of another one which *is* in the table. We have already had a rather trivial example of this when we found a primitive function of $x \longmapsto x^3$. We knew that $D(x \longmapsto x^4) = (x \longmapsto 4x^3)$, and from this we saw that $D(x \longmapsto \frac{1}{4}x^4) = (x \longmapsto x^3)$, and so a primitive of $(x \longmapsto x^3)$ is $(x \longmapsto \frac{1}{4}x^4)$. There are various rules, known as *rules of integration*, for making this kind of rearrangement. These rules correspond roughly to the rules of differentiation given in *Unit 12, Differentiation I*. Unlike the rules of differentiation, the rules of integration do not guarantee success in the sense that they do not in every case give an expression for the primitive function as a combination of known functions. They are, however, sufficient for a large proportion of the cases which commonly arise. In the end, it is the mathematician's job to hold the balance between the analytical methods, like those we discuss in this unit, and the numerical methods like those we discussed in *Unit 9, Integration I*.

We consider first the rules that arise directly from the definition of a primitive function, and then the ones that correspond to the rules for differentiating products and composite functions. All the rules of integration are collected on the reference page at the end of this text.

13.2.1 Constant Multiples, Sums

13.2.1

Discussion

We saw in *Unit 9, Integration I* and *Unit 12, Differentiation I* how to integrate and differentiate functions, in particular polynomial functions, that are constructed from simpler functions by multiplying them by constants and then adding them together.

In *Integration I* we obtained the rule for integrating a sum of two functions, say f and g:

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g,$$

Equation (1)

and also the rule for integrating a function, say f, multiplied by a number, say k:

$$\int_a^b kf = k\int_a^b f.$$

Equation (2)

We derived these rules directly from the definition of the definite integral. The Fundamental Theorem shows the rules in a new light, by relating them to the corresponding morphisms for the differentiation operator which we discussed in *Differentiation I*:

$$D(F + G) = DF + DG,$$

Equation (3)

$$D(kF) = kDF.$$

Equation (4)

Using the Fundamental Theorem, it is possible to deduce either of Equations (1) and (3) from the other, instead of proving them independently as we did in *Integration I* and *Differentiation I*. Likewise it is possible to deduce either of Equations (2) and (4) from the other. The details of these rather barren exercises are indicated in the Appendix. It is worth noting that there are corresponding rules for primitive functions too:

If F and G are primitives of continuous functions f and g, then $F + G$ is a primitive of $f + g$.

Rule 1

If F is a primitive of a continuous function f, then kF is a primitive of kf.

Rule 2

Exercise 1

Exercise 1
(3 minutes)

Prove Rule 1.

■

Exercise 2

Exercise 2
(3 minutes)

From the standard differentiation formula

$$D(x \longmapsto x^m) = x \longmapsto mx^{m-1}$$

where m is a real number and the domains of $x \longmapsto x^m$ and $x \longmapsto mx^{m-1}$ are both R^+, find primitive functions of $x \longmapsto x^5$, $x \longmapsto x^{10}$, $x \longmapsto \sqrt{x}$.

Is there any value of r for which $x \longmapsto \dfrac{x^{r+1}}{r+1}$ is not a primitive function of $x \longmapsto x^r$?

■

Exercise 3

Exercise 3
(2 minutes)

From Exercise 2 we see that the integration of $x \longmapsto \dfrac{1}{x}$ demands special attention. From the table of standard derived functions at the end of *Differentiation I*, find a primitive function of

$$x \longmapsto \frac{1}{x} \qquad (x \in R^+).$$

■

Discussion
★★

The addition and multiplication rules for integration which we have formulated so far are sufficient to integrate any polynomial function. For example the function

$$q: x \longmapsto ax^2 + bx + c \qquad (x \in R)$$

can be built up, using the two operations, multiplication by a constant and addition, from the elementary functions $x \longmapsto x^2$, $x \longmapsto x$ and $x \longmapsto 1$, which are all of the form $x \longmapsto x^m$. We saw in Exercise 2 that a primitive function of $x \longmapsto x^m$, where $m \neq -1$, is $x \longmapsto \dfrac{1}{m+1} x^{m+1}$.

It follows, by Rule 2, that:

a primitive function of $x \longmapsto ax^2$ is $x \longmapsto \frac{1}{3}ax^3$,

a primitive function of $x \longmapsto bx$ is $x \longmapsto \frac{1}{2}bx^2$,

a primitive function of $x \longmapsto c$ is $x \longmapsto cx$,

and hence, by Rule 1, that:

a primitive function of $x \longmapsto ax^2 + bx + c$ is

$$x \longmapsto \tfrac{1}{3}ax^3 + \tfrac{1}{2}bx^2 + cx.$$

Exercise 4

Exercise 4
(2 minutes)

Find all the primitive functions of

$$x \longmapsto x^5 + 3x^3 + 2x \qquad (x \in R).$$

■

Exercise 5

Exercise 5
(3 minutes)

Find a primitive function of

$$t \longmapsto 3 \sin t + 4 \cos t \qquad (t \in R).$$

■

13.2.2 Integration by Parts

13.2.2

Discussion
★★

The two remaining rules of integration are not so closely analogous to the rules of differentiation as the ones given in the preceding section, but even so they do correspond to specific rules of differentiation. In this section we formulate the rule of integration that corresponds to the rule for differentiating the product of two functions. It is useful when dealing with integrals of products of functions.

The rule for differentiating a product of two real functions f and g obtained in *Unit 12, Differentiation I*, is

$$D(f \times g) = f \times Dg + g \times Df$$

where $\times$ denotes the multiplication of functions. To convert this into a rule for integration, we take a definite integral of both sides, obtaining

$$\int_a^b D(f \times g) = \int_a^b (f \times Dg) + \int_a^b (g \times Df)$$

where a and b are any numbers such that $[a, b]$ is included in the domains of the functions Df and Dg. Applying the Fundamental Theorem of Calculus, we can put this equation into the form:

$$[f \times g]_a^b = \int_a^b (f \times Dg) + \int_a^b (g \times Df)$$

or, on rearranging,

$$\int_a^b (f \times Dg) = [f \times g]_a^b - \int_a^b (g \times Df).$$

Rule for Integration by parts
★★★

(*continued on page 23*)

Solution 13.2.1.1

Solution 13.2.1.1

To prove Rule 1 we can start either from Equation (3) or from Equation (1). Starting from Equation (3), we have

$$\begin{aligned} D(F+G) &= DF + DG \\ &= f + g, \qquad \text{by the Fundamental Theorem.} \end{aligned}$$

Therefore, $F + G$ is a primitive function of $f + g$, by the Fundamental Theorem.

Starting from Equation (1) we have, for any a and b such that $[a, b]$ lies in the domain of $f + g$,

$$\begin{aligned} \int_a^b (f+g) &= \int_a^b f + \int_a^b g \\ &= F(b) - F(a) + G(b) - G(a), \\ &\qquad \text{by the definition of a primitive function} \\ &= [F+G]_a^b. \end{aligned}$$

Therefore, $F + G$ is a primitive function of $f + g$. ■

Solution 13.2.1.2

Solution 13.2.1.2

A primitive function of $x \longmapsto x^5$ is $x \longmapsto \frac{1}{6}x^6 + c$, for any $c \in R$.
A primitive function of $x \longmapsto x^{10}$ is $x \longmapsto \frac{1}{11}x^{11} + c$, for any $c \in R$.
A primitive function of $x \longmapsto \sqrt{x} = x \longmapsto x^{1/2}$ is $x \longmapsto \dfrac{x^{3/2}}{3/2} + c$, for any $c \in R$.

There is one exceptional value of r for which the suggested primitive functions fails: it is -1. The reason is that we cannot divide by $r + 1$ if its value is zero. ■

Solution 13.2.1.3

Solution 13.2.1.3

Since

$$D(\ln) = x \longmapsto \frac{1}{x} \qquad (x \in R^+),$$

a primitive function of

$$x \longmapsto \frac{1}{x} \qquad (x \in R^+)$$

is the natural logarithm function:

$$\ln : x \longmapsto \ln x \qquad (x \in R^+).$$ ■

Solution 13.2.1.4

Solution 13.2.1.4

The set of all primitive functions of the given function consists of all the functions of the form

$$x \longmapsto \tfrac{1}{6}x^6 + \tfrac{3}{4}x^4 + x^2 + c \qquad (x \in R)$$

where c is a real constant of integration.

(You may have obtained an answer of the form

$$x \longmapsto \tfrac{1}{6}x^6 + \tfrac{3}{4}x^4 + x^2 + c_1 + c_2 + c_3 \qquad (x \in R)$$

with one constant of integration for each term in the polynomial. This is not wrong, but it is clumsy because it gives the impression that, in order to choose a primitive function from the set, we need to choose three numbers c_1, c_2 and c_3, whereas in fact only one number, $c_1 + c_2 + c_3$, is necessary.) ■

Solution 13.2.1.5

Solution 13.2.1.5

Since

$$D(t \longmapsto -3\cos t + 4\sin t) \quad \text{is} \quad t \longmapsto 3\sin t + 4\cos t,$$

a primitive function of

$$t \longmapsto 3\sin t + 4\cos t \quad \text{is} \quad t \longmapsto -3\cos t + 4\sin t. \qquad \blacksquare$$

(*continued from page 21*)

This is called the rule for integration by parts, because we integrate only *part* of the function under the integral sign on the left — the part Dg.

At first sight the rule of integration by parts does not look as though it will help much in the evaluation of integrals, because it converts one integral, $\int_a^b (f \times Dg)$, into an apparently more complicated expression that involves another integral looking very much like the one with which we started. When specific functions are used in place of the unspecified functions f and g, however, it may happen that the new integral, $\int_a^b (g \times Df)$, is easier to evaluate than the old one, $\int_a^b (f \times Dg)$, and if so, then the rule of integration by parts will have served its purpose.

As an illustration, we apply the method to the integral

$$\int_a^b x \longmapsto x\cos x.$$

The function to be integrated is the product of the function $x \longmapsto x$ and the function $x \longmapsto \cos x$, which can be abbreviated to cos, so the integral is

$$\int_a^b (x \longmapsto x) \times \cos.$$

The rule of integration by parts is

$$\int_a^b (f \times Dg) = [f \times g]_a^b - \int_a^b (g \times Df)$$

and to use it on our integral, we take f to be $x \longmapsto x$ and Dg to be cos. From the table of standard integrals we know that $D\sin = \cos$, so we take g to be the function sin. (Any other primitive function of cos would do instead, but the one used here is the natural choice, because it is the simplest.) With these choices for f and g we have

$$f : x \longmapsto x, \qquad Df : x \longmapsto 1$$

and

$$g : x \longmapsto \sin x, \qquad Dg : x \longmapsto \cos x$$

and so our integral becomes

$$\int_a^b (x \longmapsto x) \times \cos = [(x \longmapsto x) \times \sin]_a^b - \int_a^b \sin \times (x \longmapsto 1)$$

hich means the same as

$$\int_a^b x \longmapsto x\cos x = [x \longmapsto x\sin x]_a^b - \int_a^b x \longmapsto \sin x.$$

The integral on the right is easier to evaluate than the one on the left; fact it is a standard integral (see the table at the back of this text), and

$x \longmapsto -\cos x$ is a primitive function. We can therefore evaluate the right-hand side, obtaining the required integral:

$$\int_a^b x \longmapsto x\cos x = [x \longmapsto x\sin x]_a^b - [x \longmapsto -\cos x]_a^b$$

$$= b\sin b - a\sin a + \cos b - \cos a.$$

When you come to apply the formula for integration by parts you may find it more convenient to use the following form, in which the images of the functions are shown explicitly: Notation

$$\int_a^b x \longmapsto f(x) \times Dg(x)$$

$$= [x \longmapsto f(x) \times g(x)]_a^b - \int_a^b x \longmapsto g(x) \times Df(x).$$

Remember that $Dg(x)$ is the same thing as $g'(x)$, the image of x under the derived function Dg, which is the derivative of g at x. For example, this formula, when applied to the integral we have just treated, gives

$$\int_a^b x \longmapsto x\cos x = [x \longmapsto x\sin x]_a^b - \int_a^b \sin x \times (x \longmapsto 1)$$

where

$$f(x) = x, \qquad Df(x) = 1,$$

$$g(x) = \sin x, \qquad Dg(x) = \cos x.$$

The notation is, however, still clumsy, and we suggest that where:

(i) the functions have been clearly defined at the beginning of a piece of work;
(ii) there is no likelihood of confusion; for example, it is quite clear which symbol is being used for the variable defining the function;
(iii) the complexity of the work warrants it;

the "$x \longmapsto$" part of the notation of a function be dropped for the purposes of calculation. This is, of course, an "abuse of notation"; but it is normal mathematical practice to abuse notation when the situation warrants it.

Modifying our notation, we obtain:

$$\int_a^b x\cos x = [x\sin x]_a^b - \int_a^b \sin x \times 1,$$

Notation 1

and the general formula for integration of parts becomes

$$\int_a^b f(x) \times Dg(x) = [f(x) \times g(x)]_a^b - \int_a^b g(x) \times Df(x).$$

Another useful piece of notation is the following. So far we have denoted one of the primitive functions of a given function f by the corresponding capital letter F. This now becomes inconvenient, because $F \times DG$ is not a primitive of $f \times Dg$, so we denote one of the primitive functions of a given function f by

$$\int f$$

Notation 2

that is, we use the integration symbol without the end-points of integration. In terms of primitive functions, the formula for integration by parts becomes

$$\int f \times Dg = f \times g - \int g \times Df,$$

where the two primitive functions in this formula will be determined by their context: the result asserts that, if $\int g \times Df$ is one of the primitive functions of $g \times Df$, then $f \times g - \int g \times Df$ is one of the primitive functions of $f \times Dg$. For example, if we are asked to find a primitive function of $x \longmapsto x \exp x$ ($x \in R$), then we choose

$$f: x \longmapsto x, \qquad g: x \longmapsto \exp x,$$

and obtain

$$\begin{aligned}\int x \exp x &= x \exp x - \int \exp x \times 1 \\ &= x \exp x - \exp x \\ &= (x - 1) \exp x\end{aligned}$$

that is, one of the primitive functions of $x \longmapsto x \exp x$ is

$$x \longmapsto (x - 1) \exp x.$$

Exercise 1

Exercise 1
(3 minutes)

Evaluate $\int_0^\pi x \longmapsto x \sin x$.

(HINT: Take $f(x) = x$ in the formula for integration by parts.) ■

Exercise 2

Exercise 2
(3 minutes)

Apply the rule of integration by parts twice in succession to find a primitive function of $x \longmapsto x^2 \exp x$ ($x \in R$).

(HINT: take $f(x) = x^2$ in the first integration by parts.) ■

Exercise 3

Exercise 3
(3 minutes)

Apply the rule of integration by parts to the integral $\int_a^b x \cos x$ treated in the text, taking $f(x) = \cos x$ and $Dg(x) = x$. Does the rule, applied in this way, help you to evaluate the integral? What lesson do you learn from this exercise? ■

Solution 1 Solution 1

To evaluate

$$\int_0^\pi x \longmapsto x \sin x,$$

let

$f(x) = x,$ so that $Df(x) = 1;$

$g(x) = -\cos x,$ so that $Dg(x) = \sin x;$

$a = 0, \quad b = \pi.$

The formula:

$$\int_a^b f(x) \times Dg(x) = [f(x) \times g(x)]_a^b - \int_a^b g(x) \times Df(x)$$

becomes

$$\begin{aligned}\int_0^\pi x \sin x &= [-x \cos x]_0^\pi - \int_0^\pi -\cos x \\ &= -\pi \cos \pi + 0 \cos 0 + \int_0^\pi \cos x \\ &= -\pi \times (-1) + 0 + [\sin x]_0^\pi \\ &= \pi.\end{aligned}$$

■

Solution 2 Solution 2

Let

$f(x) = x^2,$ so that $Df(x) = 2x,$

$g(x) = \exp x,$ so that $Dg(x) = \exp x;$

then we obtain:

$$\int x^2 \exp x = x^2 \exp x - \int \exp x \times 2x.$$

On page 25 we found that:

$$\int x \exp x = (x - 1) \exp x.$$

Combining these results and using Equation (2) of section 13.2.1, we obtain:

$$\int x^2 \exp x = (x^2 - 2x + 2) \exp x.$$

That is, one of the primitive functions of $x \longmapsto x^2 \exp x$ is

$$x \longmapsto (x^2 - 2x + 2) \exp x.$$

■

Solution 3 Solution 3

Let

$f(x) = \cos x,$ so that $Df(x) = -\sin x,$

$g(x) = \frac{1}{2}x^2,$ so that $Dg(x) = x;$

then we obtain:

$$\int_a^b x \cos x$$

$$= [\tfrac{1}{2}x^2 \cos x]_a^b - \int_a^b -\tfrac{1}{2}x^2 \sin x.$$

This time the new integral is *more complicated* than the one we started with (the power of x in it is higher). The lesson to be learnt is that if there are several possible ways of choosing f and g in the formula for integration by parts, it is worth trying all of them if you do not at first find one that simplifies the integral. (A more advanced lesson might be that if the function to be integrated is a polynomial function times another function, it is better to make the polynomial f rather than Dg, because differentiating a polynomial reduces the degree while integrating increases it.)

■

13.2.3 An Application of Integration by Parts

13.2.3

Application

You may find this section rather difficult. It may be omitted in a first reading of this text without affecting your understanding of the subsequent material. A summary of the main steps in the argument is given at the end of the section.

As a further application of the method of integration by parts, we shall now consider a physical example.

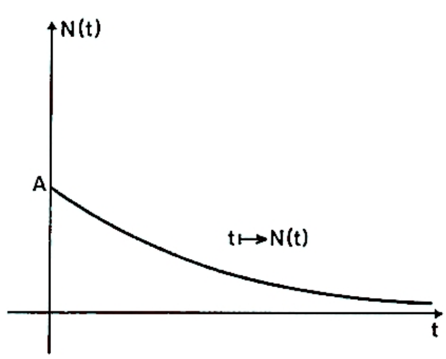
It is found experimentally that the way in which a radioactive substance, such as uranium, decays is described to a very good approximation by the formula

$$N(t) = A \exp(-ct) \qquad (t \in R^+)$$

Equation (1)

where A is a positive number, c is a positive number called the *decay constant*, and N is the function defined by

$$N: \begin{pmatrix} \text{time, } t\text{, measured} \\ \text{in years, say,} \\ \text{since some} \\ \text{arbitrary initial} \\ \text{instant} \end{pmatrix} \longmapsto \begin{pmatrix} \text{number of uranium} \\ \text{atoms remaining} \\ \text{at time } t \end{pmatrix} \qquad (t \in R^+).$$



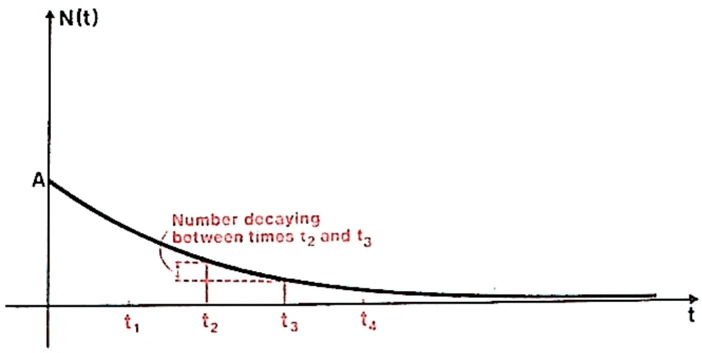
The problem is to find the *mean life-time* of the uranium atoms; that is to say, the average time a uranium atom lasts before decaying. This average can be expressed as a definite integral, which we shall evaluate using integration by parts.

The average life-time of the atoms is defined by the equation:

$$\text{average life-time} = \frac{\text{sum of the life-times of all the atoms}}{\text{number of atoms}}$$

Using the techniques developed in *Unit 9, Integration I*, we can approximate to the numerator by an integral over the time interval $[0, T]$,

where T is some very large number. We divide the time interval $[0, T]$ into m equal sub-intervals, $[0, t_1], [t_1, t_2], [t_2, t_3], \ldots, [t_{m-1}, T]$, where m is any positive integer.



The length of each sub-interval is $\frac{T}{m}$. Consider any one of these sub-intervals, $[t_{k-1}, t_k]$, say. Then the number of atoms whose times of decay lie in the interval $[t_{k-1}, t_k]$ is

$$N(t_{k-1}) - N(t_k).$$

Provided m is large, so that the interval length $\frac{T}{m} = t_k - t_{k-1}$ is small, we can make the approximation that all these atoms decay at the end of the interval, that is, at the instant t_k, so that their contribution to the sum of the life-times of all atoms decaying in the interval $[0, T]$ is approximately

$$(N(t_{k-1}) - N(t_k))t_k,$$

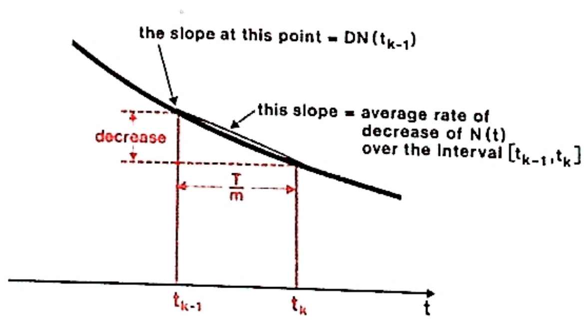
and this sum of the life-times itself is given approximately by:

$$\left\{\begin{array}{l}\text{sum of the life-times}\\ \text{of atoms decaying}\\ \text{during } [0, T]\end{array}\right\} \simeq \sum_{k=1}^{m} (N(t_{k-1}) - N(t_k))t_k. \qquad \textbf{Equation (2)}$$

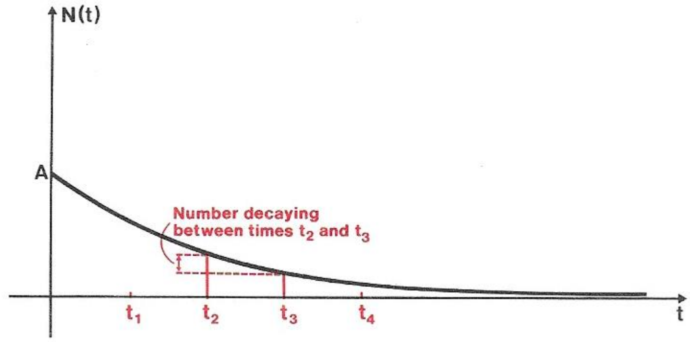
To use the definition of a definite integral of a function f (see *Unit 9, Integration I*) in the form:

$$\int_0^T f = \lim_{m \text{ large}} \frac{T}{m} \sum_{k=1}^{m} f(t_k)$$

we would like to approximate the factor $N(t_{k-1}) - N(t_k)$ in Equation (2) by something that depends only on t_k, and is proportional to $\frac{1}{m}$. Now the factor we wish to approximate, representing the number of atoms decaying during the time interval $[t_{k-1}, t_k]$, is equal to the interval length $\frac{T}{m}$ multiplied by the average rate of decrease of $N(t)$ during the interval.



where T is some very large number. We divide the time interval $[0, T]$ into m equal sub-intervals, $[0, t_1], [t_1, t_2], [t_2, t_3], \ldots, [t_{m-1}, T]$, where m is any positive integer.



The length of each sub-interval is $\frac{T}{m}$. Consider any one of these sub-intervals, $[t_{k-1}, t_k]$, say. Then the number of atoms whose times of decay lie in the interval $[t_{k-1}, t_k]$ is

$$N(t_{k-1}) - N(t_k).$$

Provided m is large, so that the interval length $\frac{T}{m} = t_k - t_{k-1}$ is small, we can make the approximation that all these atoms decay at the end of the interval, that is, at the instant t_k, so that their contribution to the sum of the life-times of all atoms decaying in the interval $[0, T]$ is approximately

$$(N(t_{k-1}) - N(t_k))t_k,$$

and this sum of the life-times itself is given approximately by:

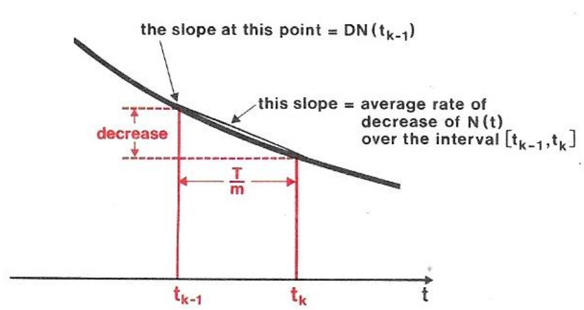
$$\left\{\begin{array}{l}\text{sum of the life-times}\\ \text{of atoms decaying}\\ \text{during } [0, T]\end{array}\right\} \simeq \sum_{k=1}^{m} (N(t_{k-1}) - N(t_k))t_k.$$

Equation (2)

To use the definition of a definite integral of a function f (see *Unit 9, Integration I*) in the form:

$$\int_0^T f = \lim_{m \text{ large}} \frac{T}{m} \sum_{k=1}^{m} f(t_k)$$

we would like to approximate the factor $N(t_{k-1}) - N(t_k)$ in Equation (2) by something that depends only on t_k, and is proportional to $\frac{1}{m}$. Now the factor we wish to approximate, representing the number of atoms decaying during the time interval $[t_{k-1}, t_k]$, is equal to the interval length $\frac{T}{m}$ multiplied by the average rate of decrease of $N(t)$ during the interval.



For g we need a function whose derived function is $t \longmapsto \exp(-ct)$. In *Unit 12, Differentiation I* we saw that the exponential function is its own derived function; this suggests trying $t \longmapsto \exp(-ct)$ for g. In fact the derived function of $t \longmapsto \exp(-ct)$ is $t \longmapsto -c\exp(-ct)$, which is not quite the Dg that we want; but by the constant factor rule we can remove the unwanted factor $(-c)$ by taking

$$g : t \longmapsto \frac{\exp(-ct)}{-c}$$

instead. Substituting for $f(t)$ and $g(t)$ in the integration-by-parts formula, with $a = 0$ and $b = T$, we find:

$$\begin{aligned}\int_0^T ct\exp(-ct) &= \left[ct\frac{\exp(-ct)}{-c}\right]_0^T - \int_0^T \frac{\exp(-ct)}{-c}c\\ &= [-t\exp(-ct)]_0^T - \frac{1}{c}[\exp(-ct)]_0^T\\ &= -T\exp(-cT) + 0 - \frac{\exp(-cT)}{c} + \frac{1}{c}\end{aligned}$$

so that, by Equation (6)

$$\begin{Bmatrix}\text{average}\\ \text{life-time}\end{Bmatrix} = \frac{1}{c} + \lim_{T\text{ large}}\left\{-T\exp(-cT) - \frac{1}{c}\exp(-cT)\right\}.$$

Equation (7)

This disposes of the integration.

The last step is to deal with the limit in Equation (7).

The graph of the function N shows that the term $\frac{1}{c}\exp(-cT)$ has limit 0 for large t. The limit of the other term is not quite so obvious, because the small quantity $\exp(-cT)$ is multiplied by a factor T which is large, so that it is not immediately clear whether their product is large or small. Calculation shows, however, that the product is very small for large T, as can be seen from the table at the right; and in fact it is possible to prove (for $c > 0$) that

$$\lim_{T\text{ large}}(cT\exp(-cT)) = 0.$$

$x\,(= cT)$	$x\exp(-x)$
0	0
1	0.368
2	0.257
3	0.149
4	0.073
5	0.034
6	0.015
7	0.006
8	0.003
9	0.001
10	0.000

Thus each term in the limit in Equation (7) has limit zero, and the formula reduces to:

$$\text{average life-time} = \frac{1}{c}.$$

This answers the problem posed at the beginning of this section.

Summary

Summary

1 Number of uranium atoms remaining at time t is given by

$$N(t) = A\exp(-ct) \qquad (t \in R^+).$$

Equation (1)

2 Average life-time of atoms is defined by

$$\text{average life-time} = \frac{\text{sum of the life-times of all the atoms}}{\text{number of atoms}}$$

3 Number of atoms decaying in time interval $[t_{k-1}, t_k]$ is

$$N(t_{k-1}) - N(t_k),$$

and if we assume that they all decay at the instant t_k, the total life-time of these atoms is, approximately,

$$(N(t_{k-1}) - N(t_k))t_k.$$

4 We divide the interval $[0, T]$, where T is large, into m equal sub-intervals. The total life time of all atoms decaying in this interval is, approximately,

$$\sum_{k=1}^{m} (N(t_{k-1}) - N(t_k))t_k$$

$$= \frac{T}{m}\sum_{k=1}^{m}\left(\frac{N(t_{k-1}) - N(t_k)}{t_k - t_{k-1}}\right)t_k, \quad \text{since } \frac{T}{m} = t_k - t_{k-1}.$$ Equations (2) and (3)

$$\simeq \frac{T}{m}\sum_{k=1}^{m}(-t_k DN(t_k)).$$ Equation (4)

5 In the limit, for very large m,

$$\left\{\begin{array}{l}\text{sum of life-times}\\ \text{of atoms decaying}\\ \text{during } [0, T]\end{array}\right\} = \int_0^T t \longmapsto -t\,DN(t)$$

and so

$$\left\{\begin{array}{l}\text{sum of the life-times}\\ \text{of all the atoms}\end{array}\right\} = \lim_{T\,\text{large}} \int_0^T t \longmapsto -t\,DN(t)$$

$$\left\{\begin{array}{l}\text{average}\\ \text{life-time}\end{array}\right\} = \lim_{T\,\text{large}} \frac{1}{A}\int_0^T t \longmapsto -t\,DN(t)$$ Equation (5)

$$= \lim_{T\,\text{large}} \int_0^T t \longmapsto ct\exp(-ct)$$ Equation (6)

$$= \frac{1}{c} + \lim_{T\,\text{large}}\left(-T\exp(-cT) - \frac{1}{c}\exp(-cT)\right)$$ Equation (7)

$$= \frac{1}{c}.$$

13.2.4 Integration by Substitution

13.2.4

Discussion

In the previous sections, we have investigated the effect of the mapping I on the arithmetic operations defined on functions. As you may recall from *Unit 1, Functions*, there is another operation defined on functions. In this section we shall discuss how the integration mapping affects *composite functions*. As with the arithmetic operations, we can save ourselves a lot of work by using the results we obtained in *Unit 12, Differentiation I* for differentiating composite functions. We saw in *Unit 1* that when inverting a function it is often useful to think of it as the composition of simpler functions. Similarly the concept of composition is useful in the context of integration. One can often evaluate an integral of a function most easily by finding a primitive which is a composition of more elementary functions. First of all we must find out how to integrate a composite function. As an illustration, consider the problem of evaluating

$$\int_a^b x \longmapsto x\cos(x^2)$$

where a and b are positive real numbers. This looks very similar to the integral which we evaluated by parts in the preceding two sections, but

the fact that the integrand now involves $\cos(x^2)$ instead of $\cos x$ makes a big difference. If we try to apply the method that we used for

$$\int_a^b x \longmapsto x \cos x,$$

we find that the functions f and g enter the calculation, where

$$f(x) = x \qquad Dg(x) = \cos(x^2)$$
$$Df(x) = 1, \qquad g(x) = ?$$

Before, we had $Dg(x) = \cos x$, so that $g(x)$ was $\sin x$; but in this case there is no simple function having derived function $x \longmapsto \cos(x^2)$ to use for g. This is not the only way of using the rule of integration by parts here, but none of the alternatives is much help either; so instead of labouring the integration by parts method any more, let us look instead at the integral

$$\int_a^b x \longmapsto x \cos(x^2)$$

from a fresh point of view.

One way to evaluate the integral would be to find a suitable primitive function, and by the Fundamental Theorem of Calculus this primitive function, F say, will satisfy the equation:

$$DF(x) = x\cos(x^2) \qquad (x \in [a, b]). \qquad \text{Equation (1)}$$

The expression $\cos(x^2)$ suggests that F may have the form

$$F(x) = G(x^2) \qquad (x \in [a, b]) \qquad \text{Equation (2)}$$

where G is some new function, to be chosen in accordance with Equation (1). To use this equation we differentiate the function in Equation (2), obtaining

$$DF(x) = 2x\, DG(x^2) \qquad (x \in [a, b])$$

by the rule for differentiating composite functions (see *Unit 12*, section 12.2.5); Equation (1) then gives:

$$2x\, DG(x^2) = x\cos(x^2) \qquad (x \in [a, b]).$$

The natural way to satisfy this condition is to make

$$DG(x^2) = \tfrac{1}{2}\cos(x^2) \qquad (x \in [a, b]),$$

that is,

$$DG(u) = \tfrac{1}{2}\cos u \qquad (u \in [a^2, b^2]) \qquad \text{Equation (3)}$$

where u stands for x^2. We have now reduced the problem of finding a primitive of $x \longmapsto x\cos(x^2)$ to the simpler one of finding a primitive function of $u \longmapsto \frac{1}{2}\cos u$.

By the table of standard integrals, this latter primitive function is $\frac{1}{2}\sin$, so from Equation (3) we obtain:

$$G(u) = \tfrac{1}{2}\sin u \qquad (u \in [a^2, b^2]),$$

and then Equation (2) gives a required primitive function F, where

$$F(x) = \tfrac{1}{2}\sin(x^2) \qquad (x \in [a, b]).$$

The integral we set out to evaluate is therefore

$$\int_a^b x\cos(x^2) = [\tfrac{1}{2}\sin(x^2)]_a^b$$
$$= \tfrac{1}{2}\sin(b^2) - \tfrac{1}{2}\sin(a^2).$$

Exercise 1

Exercise 1
(3 minutes)

By filling in the empty boxes in the working which follows, evaluate $\int_{\pi^2/4}^{\pi^2} x \longmapsto \dfrac{\sin\sqrt{x}}{\sqrt{x}}$.

If

$$F(x) = G(\sqrt{x}) \qquad \left(x \in \left[\frac{\pi^2}{4}, \pi^2\right]\right),$$

then we have

$$DF(x) = \boxed{}\, DG(\sqrt{x}). \qquad \text{(i)}$$

Putting

$$DF(x) = \frac{\sin\sqrt{x}}{\sqrt{x}}, \quad \text{we get}$$

$$\sin\sqrt{x} = \boxed{}\,. \qquad \text{(ii)}$$

If we put

$$\sqrt{x} = u, \quad \text{then}$$

$$DG(u) = \boxed{} \qquad \text{(iii)}$$

and a primitive function for $u \longmapsto 2\sin u$ is

$$u \longmapsto \boxed{}\,. \qquad \text{(iv)}$$

Thus we find

$$G(u) = \boxed{}\,. \qquad \text{(v)}$$

The definition of G gives:

$$G(u) = G(\sqrt{x}) = F(x).$$

Thus we can find $F(x)$ in terms of x. The expression is

$$F(x) = \boxed{} \qquad \text{(vi)}$$

and since

$$\int_0^{\pi^2} x \longmapsto \frac{\sin\sqrt{x}}{\sqrt{x}} = [F]_{\pi^2/4}^{\pi^2}$$

we have

$$\int_{\pi^2/4}^{\pi^2} x \longmapsto \frac{\sin\sqrt{x}}{\sqrt{x}} = \boxed{} \qquad \text{(vii)}$$

Solution 1 Solution 1

(i) $DF(x) = \frac{1}{2\sqrt{x}} G(\sqrt{x})$.

(ii) $\sin \sqrt{x} = \frac{1}{2} DG(\sqrt{x})$.

(iii) $DG(u) = 2 \sin u$.

(iv) $u \longmapsto -2 \cos u$.

(v) $G(u) = -2 \cos u + c$, where c is any constant.

(vi) $F(x) = -2 \cos \sqrt{x} + c$.

(vii) $$[x \longmapsto -2 \cos \sqrt{x}]_{\pi^2/4}^{\pi^2} = -2 \cos \pi + 2 \cos \frac{\pi}{2}$$
$$= 2.$$

Exercise 2 Exercise 2 (4 minutes)

By writing $F(x) = G(-x^2)$, and making a suitable choice for F, evaluate

$$\int_a^b x \longmapsto x \exp(-x^2),$$

where a and b are any positive real numbers such that $a < b$. ■

Main Text

To make the application of this method as convenient as possible, it is worth setting up a general formula, just as we did for integration by parts, embodying the steps that are common to every application of the method. The method we have been discussing in this section comes from the result for differentiating a composite function. For example, in the evaluation of

$$\int_a^b x \longmapsto x \cos(x^2)$$

we looked for a primitive function F in the form

$$F(x) = G(x^2)$$

(see Equation (2)). In general, this composite function has the form

$$F = G \circ k,$$

in the notation introduced in *Unit 1*, so

$$F(x) = G(k(x)).$$

Mappings and Functions

In general, if the integral we are trying to evaluate is $\int_a^b f$, then the Fundamental Theorem of Calculus tells that $f = DF$, and hence, by the rule for differentiating composite functions (see *Unit 12*, section 12.2.5), we have:

$$F' = (G' \circ k) \times k'$$

that is,

$$f = DF = (DG \circ k) \times Dk.$$

Thus the integral we are trying to evaluate has the form

$$\int_a^b f = \int_a^b (DG \circ k) \times Dk$$ Equation (4)

and its value is

$$F(b) - F(a) = G(k(b)) - G(k(a)).$$ Equation (5)

Since Equation (4) will not give us G directly, but will give us DG (if we know k), it is best to express Equation (5) in terms of DG too; by the Fundamental Theorem we can put Equation (5) in the form:

$$F(b) - F(a) = \int_{k(a)}^{k(b)} DG.$$

Equation (6)

Writing g for DG and combining Equations (4) and (6), we get:

$$\int_a^b (g \circ k) \times Dk = \int_{k(a)}^{k(b)} g.$$

Rule for Integration by Substitution
* * *

This is the basic rule for integration by substitution. It is repeated in the table of rules of integration accompanying this text. Do not attempt to memorize the formula: it is much better to look it up when you need it.

Example 1

Example 1

As an example, we apply the rule to the integral we considered at the beginning of this section, that is:

$$\int_a^b x \longmapsto x \cos(x^2).$$

Our previous calculation corresponds to the choice

$$k(x) = x^2 \qquad (x \in R).$$

Since $Dk(x) = 2x$, and we require

$$(g \circ k) \times Dk = x \longmapsto x \cos(x^2),$$

we take

$$g \circ k(x) = \tfrac{1}{2}\cos(x^2) \qquad (x \in R)$$

which is the equivalent to

$$g(u) = \tfrac{1}{2}\cos u \qquad (u \in R, \text{ and } u \geqslant 0)$$

where we have written u for $k(x)$, that is, for x^2.

This substitution of u for $k(x)$ greatly simplifies the manipulations, and accounts for the name "integration by substitution". The rule now tells us that:

$$\begin{aligned}\int_a^b x \longmapsto \cos(x^2) &= \int_{k(a)}^{k(b)} g \\ &= \int_{a^2}^{b^2} u \longmapsto \tfrac{1}{2}\cos u \\ &= [\tfrac{1}{2}\sin u]_{a^2}^{b^2} \\ &= \tfrac{1}{2}\sin(b^2) - \tfrac{1}{2}\sin(a^2)\end{aligned}$$

as we found before. ∎

Exercise 3

Exercise 3
(5 minutes)

Evaluate $\int_0^\pi x \longmapsto \sin(3x)$ using the rule for integration by substitution with $k(x) = 3x$ $(x \in R)$. ∎

Solution 2 Solution 2

Let F be a primitive function of $x \longmapsto x\exp(-x^2)$; as suggested, we guess that is has the form

$$F(x) = G(-x^2) \qquad (x \in [a, b]).$$

Then

$$DF(x) = -2xDG(-x^2) \qquad (x \in [a, b])$$

that is,

$$x\exp(-x^2) = -2xDG(-x^2) \qquad (x \in [a, b]),$$

or

$$DG(u) = -\tfrac{1}{2}\exp(u) \qquad (u \in [a^2, b^2]),$$

where

$$u = -x^2.$$

Now a primitive function for $u \longmapsto -\frac{1}{2}\exp(u)$ is

$$u \longmapsto -\tfrac{1}{2}\exp(u) \qquad (u \in [a^2, b^2]),$$

so we can take

$$F(x) = -\tfrac{1}{2}\exp(-x^2) \qquad (x \in [a, b]),$$

and find

$$\int_a^b x \longmapsto x\exp(-x^2) = -\tfrac{1}{2}\exp(-b^2) + \tfrac{1}{2}\exp(-a^2). \qquad \blacksquare$$

Solution 3 Solution 3

Applying the rule with $a = 0$ and $b = \pi$, we obtain:

$$\int_0^\pi (g \circ k) \times Dk = \int_{k(0)}^{k(\pi)} g.$$

The integrand is

$$g(k(x)) \times Dk(x) = \sin(3x),$$

and we have $Dk(x) = 3$; so we want

$$3g(k(x)) = \sin(3x),$$

that is, $g(u) = \frac{1}{3}\sin u$, where $u = 3x$.

The given integral is therefore equal to

$$\begin{aligned}\int_{k(0)}^{k(\pi)} \tfrac{1}{3}\sin u &= [-\tfrac{1}{3}\cos u]_0^{3\pi} \\ &= -\tfrac{1}{3}\cos 3\pi + \tfrac{1}{3}\cos 0 \\ &= \tfrac{1}{3} + \tfrac{1}{3} \\ &= \tfrac{2}{3}.\end{aligned} \qquad \blacksquare$$

Sometimes the rule of substitution is most conveniently used in a "backwards" form in which we start from the right-hand side of the basic formula

Discussion

$$\int_a^b (g \circ k) \times Dk = \int_{k(a)}^{k(b)} g$$

instead of the left-hand side. In this case, with $\int_{k(a)}^{k(b)} g$ given, we know $k(a)$ and $k(b)$ and wish to find a and b. That is, we want to *invert* the function

This is possible if k is one–one. Writing α for $k(a)$ and β for $k(b)$ we then have the rule:

$$\int_\alpha^\beta g = \int_{h(\alpha)}^{h(\beta)} (g \circ k) \times Dk$$

Equation (7)

where h is the inverse of the function k.

Example 2

Example 2

Evaluate $\int_{-1}^{1} x \longmapsto \sqrt{1+x}$ using $u = \sqrt{1+x}$. Here we have:

$$\alpha = -1, \qquad \beta = 1$$

$$g(x) = \sqrt{1+x} \qquad (x \in [-1, 1])$$

$$u = \sqrt{1+x} \qquad (x \in [-1, 1]).$$

Notice that in this "backwards" form of the rule, it is h (the inverse of k) that maps x to u; so we choose:

$$h(x) = u = \sqrt{1+x} \qquad (x \in [-1, 1])$$

which gives, on inverting this function,*

$$k(u) = u^2 - 1 \qquad (u \in [0, \sqrt{2}]).$$

Substitution in Equation (7) gives:

$$\begin{aligned}\int_{-1}^{1} x \longmapsto \sqrt{1+x} &= \int_{h(-1)}^{h(1)} (u \longmapsto g(u^2 - 1)) \times Dk(u)\\ &= \int_0^{\sqrt{2}} (u \longmapsto u) \times 2u\\ &= [\tfrac{2}{3}u^3]_0^{\sqrt{2}}\\ &= \tfrac{4}{3}\sqrt{2}.\end{aligned}$$

■

Exercise 4

Exercise 4
(5 minutes)

Evaluate $\int_0^1 x \longmapsto x\sqrt{1-x^2}$ using Equation (7), with $u = \sqrt{1-x^2}$ (that is, $h(x) = \sqrt{1-x^2}$).

■

* We have

$$(h(x))^2 = 1 + x$$

so

$$x = (h(x))^2 - 1.$$

Solution 4 Solution 4

Here, $u = h(x) = \sqrt{1 - x^2}$ ($x \in [0, 1]$), and h is a one-one function for this domain, so that it has an inverse given by

$$x = k(u) = \sqrt{1 - u^2} \qquad (u \in [0, 1]),$$

where $h(1) = 0$ and $h(0) = 1$.

Thus:

$$\int_0^1 x \longmapsto x\sqrt{(1 - x^2)}$$

$$= \int_1^0 (u \longmapsto \sqrt{1 - u^2} \times \sqrt{1 - (1 - u^2)}) \times \frac{-u}{\sqrt{1 - u^2}}$$

$$= \int_1^0 u \longmapsto -u^2$$

$$= [-\tfrac{1}{3}u^3]_1^0$$

$$= \tfrac{1}{3}.$$

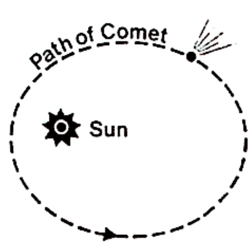
■

13.2.5 An Application of Integration by Substitution

13.2.5

Application

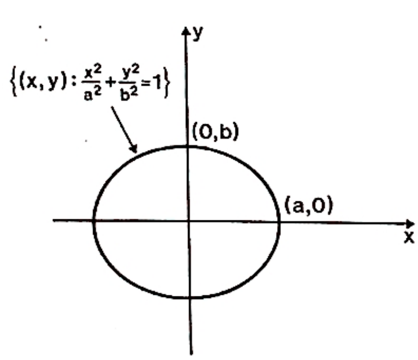
As a further illustration of the rule of integration by substitution, let us apply it to the problem of calculating the area enclosed by an ellipse, the curve giving the shape of the orbit of a planet or comet moving under the gravitational influence of the sun.



For our purposes the ellipse may be defined as the graph of the equation

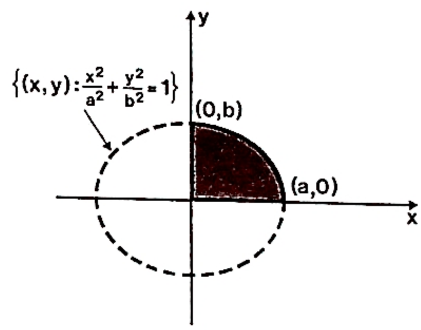
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$ Equation (1)

where x and y are Cartesian co-ordinates, and a and b are positive real numbers.*



* Note that a and b are being used here in a different sense from that used earlier in this text.

We can exploit the symmetry of the ellipse and use the techniques of *Unit 9, Integration I*. The co-ordinate axes cut the ellipse into four congruent parts, and the total area of the ellipse is just four times the area of any one of them, say the part for which $x \geqslant 0$ and $y \geqslant 0$.



This quarter-ellipse is the type of area that we can express as an integral. The formula for the area is:

$$\text{area of quarter-ellipse} = \int_0^a f.$$

Here f is the function whose graph is the part of the ellipse shown in the figure. The domain of this function is $[0, a]$ and in view of Equation (1) it must satisfy

$$\frac{x^2}{a^2} + \frac{f(x)^2}{b^2} = 1 \qquad (x \in [0, a]),$$

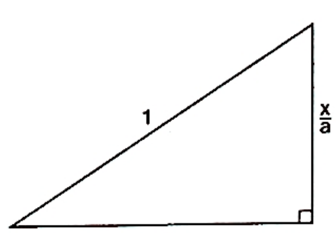
or, solving for $f(x)$,

$$f(x) = b\sqrt{1 - \frac{x^2}{a^2}} \qquad (x \in [0, a]).$$

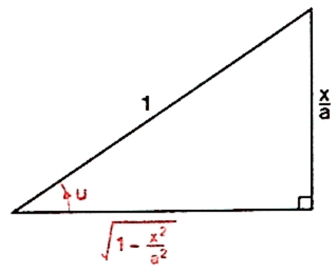
(The negative square root $f(x) = -b\sqrt{1 - \dfrac{x^2}{a^2}}$ would also be a solution, but would give the wrong part of the ellipse — the part below the x-axis.) We now have:

$$\text{area of quarter-ellipse} = \int_0^a x \longmapsto b\sqrt{1 - \frac{x^2}{a^2}}.$$

The substitution (the choice of the function $h: x \longmapsto u$) which enables us to evaluate this integral is not as obvious as the ones we used in the last section. However, the presence of the square root sign with the square of $\dfrac{x}{a}$ inside it suggests the use of Pythagoras' theorem, applied to the triangle shown below.



Pythagoras' theorem tells us that the base of this triangle is $\sqrt{1 - \frac{x^2}{a^2}}$, which is the unpleasant part of the integral.



If we call the angle at the left of this triangle u then we have:

$$\sin u = \frac{x}{a},$$

$$\cos u = \sqrt{1 - \frac{x^2}{a^2}},$$

so that there is a chance of simplifying the integral by the substitution of $a \sin u$ for x. The rule for integration by substitution, with the end-points altered to conform to the notation of this section, is:

$$\int_0^a x \longmapsto f(x) = \int_{h(0)}^{h(a)} (u \longmapsto f(k(u))) \times Dk(u)$$ Equation (2)

where

$$h : x \longmapsto u \qquad (x \in [0, a]),$$

$$k : u \longmapsto x \qquad (u \in [h(0), h(a)]).$$

The substitution $\sin u = \frac{x}{a}$ corresponds to:

$$k(u) = a \sin u$$

$$h(x) = \text{the angle in the first quadrant, with } \sin(h(x)) = \frac{x}{a}.$$

The triangle shows that $h(x)$ increases with x for $x \in [0, a]$, and therefore h is a one–one function; it also shows that when $x = 0$, then $u = 0$, and when $x = a$, then $u = \frac{\pi}{2}$; that is,

$$h(0) = 0$$

$$h(a) = \frac{\pi}{2}.$$

Substituting this information about h and k into Equation (2), we obtain:

$$\int_0^a x \longmapsto b\sqrt{1 - \frac{x^2}{a^2}} = \int_0^{\pi/2} (u \longmapsto b \cos u) \times (a \cos u)$$

since $\sqrt{1 - \frac{x^2}{a^2}} = \cos u$, and $Dk(u) = a \cos u$.

This integral simplifies to:

$$ab \int_0^{\pi/2} u \longmapsto \cos^2 u.$$

This is still not a standard integral, but at least we have got rid of the square root.

To complete the evaluation of the integral, we should like to use the standard forms for integrating sines and cosines. This is not immediately possible because the cosine is squared, so the first step is to express $\cos^2 u$ in terms of a cosine that is not squared. It is one of the very useful properties of the trigonometric functions that this is possible. We use the identities*

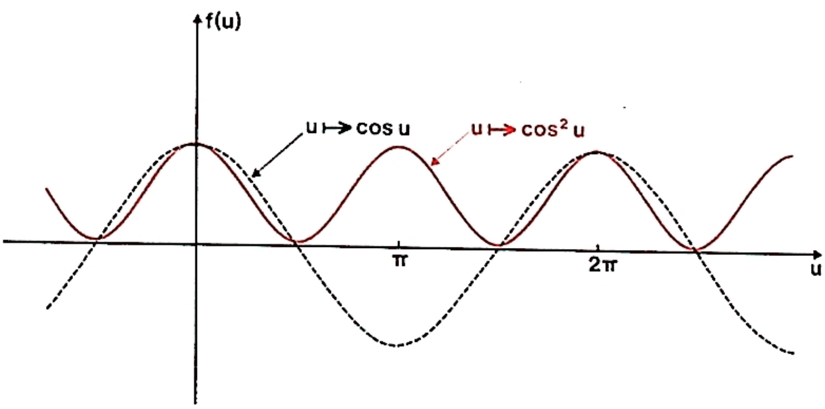
(See RB 10)

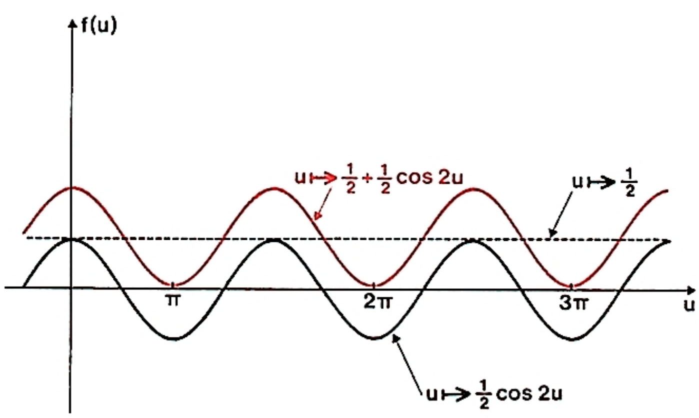
$$\left.\begin{aligned} 1 &= \cos^2 u + \sin^2 u \\ \cos 2u &= \cos^2 u - \sin^2 u \end{aligned}\right\} \qquad (u \in R).$$

Adding and dividing by 2 gives

$$\cos^2 u = \tfrac{1}{2} + \tfrac{1}{2}\cos 2u \qquad (u \in R)$$

Equation (3)





Exercise 1

Exercise 1
(5 minutes)

Complete the evaluation of the integral, and hence of the area of the ellipse. Verify your formula by considering the special case when $a = b$. ■

* An *identity* (in this context) is a formula, such as $f(x) = g(x)$, that connects images under two functions and holds for all elements in their common domain (as opposed to an equation, which holds for only a few special values).

Substituting from Equation (3) into the integral, and then using the rules for sums and constant factors, we obtain:

$$\tfrac{1}{2}ab \int_0^{\pi/2} u \longmapsto 1 + \tfrac{1}{2}ab \int_0^{\pi/2} u \longmapsto \cos 2u.$$

The first integral is now a standard integral and the second is almost in standard form; the simplest way to evaluate it is to note that

$$D(u \longmapsto \sin 2u) = u \longmapsto 2 \cos 2u$$

so that $D(u \longmapsto \tfrac{1}{2} \sin 2u) = u \longmapsto \cos 2u$ and hence $u \longmapsto \tfrac{1}{2} \sin 2u$ is a suitable primitive.

The integral thus becomes

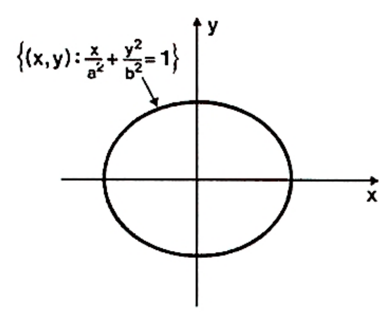
$$\tfrac{1}{2}ab[u \longmapsto u]_0^{\pi/2} + \tfrac{1}{2}ab[u \longmapsto \tfrac{1}{2} \sin 2u]_0^{\pi/2} = \tfrac{1}{4}\pi ab + 0.$$

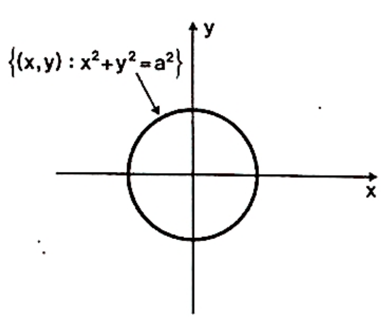
This is the area of the quarter-ellipse; so we conclude that

$$\text{area of ellipse} = \pi ab$$

(We are not suggesting that this is the best way to calculate the area of an ellipse; if you have a feeling for geometry you may like to try to think of a more obvious method. Our main purpose here is to illustrate the method of integration by substitution.)

The check is to consider the special case where $b = a$, in which case the ellipse is a circle of radius a, whose area is correctly given by the formula above as πa^2.





■

13.2.6 The Evaluation of π

13.2.6

Application

In addition to giving us the area of an ellipse of known dimensions in terms of the number π, the integrals we have been studying can also be used to provide a method for calculating π numerically. We have shown in fact that

Equation (1)

$$\int_0^a x \longmapsto b\sqrt{1-\frac{x^2}{a^2}} = \tfrac{1}{4}\pi ab;$$

so by choosing convenient values for a and b (say $a = b = 1$) and evaluating the integral by Simpson's rule* or some other numerical method, we can, in principle, calculate π.

When x is close to a, the slope of the graph of $x \longmapsto \sqrt{1-\frac{x^2}{a^2}}$ becomes very large and the approximating strips used in Simpson's rule are nothing like rectangles. Thus Equation (1) does not lead to a very accurate method for calculating π. There are other integrals which yield better results. The simplest is

Equation (2)

$$\int_0^1 x \longmapsto \frac{1}{1+x^2} = \frac{\pi}{4}.$$

Exercise 1

Exercise 1
(5 minutes)

Verify the result in Equation (2), using the substitution

$$x = \tan u \qquad (x \in [0, 1]),$$

with u an angle in the first quadrant. You will need to use the identity

$$1 + \tan^2 u = \sec^2 u \qquad (u \in R).$$

■

Exercise 2

Exercise 2
(5 minutes)

Calculate a value for π by applying Simpson's rule with four strips to the tegral in Equation (2).

Simpson's rule for four strips is

$$\int_a^b f = \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4)$$

here h is the interval width and $y_0, y_1, \ldots, y_4$ are the ordinates at the ends of the intervals.

Work to four places of decimals. Part of a table of reciprocals is given at the right.

■

x	$\frac{1}{x}$
...	...
1.06	0.9434
1.07	0.9346
...	...
1.56	0.6410
1.57	0.6369
...	...

* See *Unit 9, Integration I*, section 9.4.2.

Solution 1

Solution 1

The rule of integration by substitution tells us that

$$\int_0^1 x \longmapsto \frac{1}{1+x^2} = \int_{h(0)}^{h(1)} \left(u \longmapsto \frac{1}{1+k(u)^2}\right) \times Dk(u).$$

Taking $x = \tan u$, we have:

$$k : u \longmapsto \tan u \qquad \left(u \in \left[0, \frac{\pi}{4}\right]\right),$$

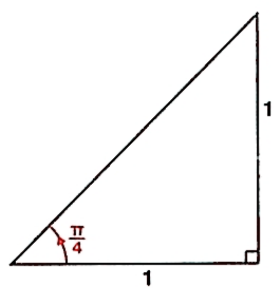
$$h : x \longmapsto \text{the angle in } \left[0, \frac{\pi}{4}\right] \text{ whose tangent is } x$$

$$(x \in R \text{ and } x \geqslant 0),$$

and, in particular, since we have restricted u to the first quadrant:

$$h(0) = 0$$

$$h(1) = \frac{\pi}{4}, \quad \text{since} \quad \tan\frac{\pi}{4} = 1.$$



The integration therefore gives:

$$\begin{aligned}\int_0^1 x \longmapsto \frac{1}{1+x^2} &= \int_0^{\pi/4} \left(u \longmapsto \frac{1}{1+\tan^2 u}\right) \times \sec^2 u \\ &= \int_0^{\pi/4} u \longmapsto 1 \\ &= [u \longmapsto u]_0^{\pi/4} \\ &= \frac{\pi}{4}.\end{aligned}$$

■

Solution 2

Solution 2

Using linear interpolation in the table of reciprocals (see *Unit 4, Finite Differences*) we find:

x	$1 + x^2$	$\frac{1}{1+x^2}$
0	1.0000	1.0000
0.25	1.0625	0.9412
0.5	1.2500	0.8000
0.75	1.5625	0.6400
1.00	2.0000	0.5000

By Simpson's rule, we find (since the interval width is 0.25)

$$\frac{\pi}{4} = \int_0^1 x \longmapsto \frac{1}{1+x^2}$$

$$\simeq \frac{0.25}{3}(1.0000 + 4 \times 0.9412 + 2 \times 0.8000$$

$$+ 4 \times 0.6400 + 0.5000)$$

$$= \frac{0.25}{3}(9.4248);$$

so

$$\pi \simeq \tfrac{1}{3}(9.4248) = 3.1416,$$

which is a surprisingly accurate result from so little work! ■

13.2.7 The Leibniz Notation

13.2.7

Notation

In the Mathematics Foundation Course we use the notation $\int_a^b f$ for a definite integral, because it stresses the fact that the integrand is a *function*. We chose this notation in the hope that it would make the fundamental ideas as clear as possible, by exhibiting the relation of our various theorems and techniques to the fundamental ideas about functions (addition, multiplication and composition of functions), introduced in *Unit 1*. However, many people (and this includes many authors of calculus text books) find it convenient to use a different notation which is due to Leibniz.

The Leibniz notation was introduced briefly in *Unit 9, Integration I*, and mentioned also in *Unit 12, Differentiation I*. If $y = f(x)$, then, in the Leibniz notation,

$$\int_a^b y\,dx \qquad \text{corresponds to} \qquad \int_a^b f$$

$$[y]_a^b \quad \text{or} \quad [y]_{x=a}^{x=b} \qquad \text{corresponds to} \qquad [f]_a^b$$

and

$$\int y\,dx \qquad \text{corresponds to} \quad F(x) \quad \text{or} \quad \left(\int f\right)(x)$$

where F is one of the primitive functions of f. Thus

$$\int_0^1 \frac{1}{1+x^2}\,dx \quad \text{means} \quad \int_0^1 x \longmapsto \frac{1}{1+x^2}.$$

The formula on the left would normally be abbreviated to

$$\int_0^1 \frac{dx}{1+x^2}.$$

To transcribe a formula from Leibniz notation to function notation, or vice versa, you need only remember that:

$$\int_a^b y\,dx \quad \text{corresponds to} \quad \int_a^b x \longmapsto y,$$

where y may be replaced by any expression in x, such as $\frac{1}{1+x^2}$ or $f(x)$,

and that

$$\frac{dy}{dx} \quad \text{corresponds to } Df(x).$$

where f is defined by $f: x \longmapsto y$, with a suitable domain (see *Differentiation I*). It is important to realize that $\int_a^b y\,dx$ depends on a, b and the function $x \longmapsto y$, while $\int y\,dx$ gives the image of x under some function — it depends on the value of x, on the constant of integration, and on the function $x \longmapsto y$. For example, we have

$$\int_a^b x\,dx = \tfrac{1}{2}b^2 - \tfrac{1}{2}a^2$$

while

$$\int x\,dx = \tfrac{1}{2}x^2 + c$$

where c is some real number.

Our "abuse of notation" (see section 13.2.2, page 24) gives expressions which are very close to the corresponding expressions in the Leibniz notation.

Exercise 1

Exercise 1
(3 minutes)

Transcribe the following expressions from the Leibniz notation to the "unabused" function notation.

(i) $\int_{-1}^{1} (1 + x)\,dx$,

(ii) $\int (1 - x)\,dx$,

(iii) $\int_a^b f(x)\,dx = \left[\int f(x)\,dx\right]_a^b$,

(iv) $\frac{d}{db}\int_a^b f(x)\,dx = f(b)$. ■

Exercise 2

Exercise 2
(5 minutes)

Transcribe the rules of integration from the forms:

(i) $\int_a^b kf = k\int_a^b f \qquad (k \in R)$,

(ii) $\int_a^b (f + g) = \int_a^b f + \int_a^b g$,

(iii) $\int_a^b f \times Dg = [f \times g]_a^b - \int_a^b g \times Df$,

(iv) $\int_a^b (g \circ k) \times Dk = \int_{k(a)}^{k(b)} g$,

(v) $\int_\alpha^\beta g = \int_{h(\alpha)}^{h(\beta)} (g \circ k) \times Dk$, where h is the inverse function of k,

into Leibniz notation, and write down an example of each formula with specific numbers and functions in place of f, g, h, etc. ■

Discussion

The Leibniz forms of the rules of integration can be simplified by introducing new variables to stand for $f(x)$, $g(x)$, etc.

Defining $p = f(x)$, $q = g(x)$ and (as we did in section 13.2.4) $u = k(x)$ or $h(x)$, we can write the rules as follows:

(i) $$\int_a^b kp\,dx = k\int_a^b p\,dx.$$

(ii) $$\int_a^b (p+q)\,dx = \int_a^b p\,dx + \int_a^b q\,dx.$$

(iii) $$\int_a^b p\frac{dq}{dx}dx = [pq]_a^b - \int_a^b q\frac{dp}{dx}dx.$$

(iv) $$\int_a^b g(u)\frac{du}{dx}\,dx = \int_{k(a)}^{k(b)} g(u)\,du.$$

(v) $$\int_a^b q\,dx = \int_{h(a)}^{h(b)} q\frac{dx}{du}\,du.$$

The transformation of the integrand in (iv) and (v) is easily remembered since the essential step is embodied in the equation

$$dx = \frac{dx}{du}du$$

(but don't forget that in each of (iv) and (v) the integrals on the two sides of the equation have different end-points).

To transcribe in the opposite direction, the crucial point is to work always with the *images* under the functions, not the functions themselves. For example, to convert $\int_a^b f \times g$ into Leibniz notation, we first express it in terms of images: thus

$$\int_a^b f \times g = \int_a^b x \longmapsto f(x)g(x),$$

which is $\int_a^b f(x)g(x)\,dx$ in the Leibniz notation.

Solution 1

Solution 1

(i) $\int_{-1}^{1} x \longmapsto 1 + x;$

(ii) $F(x)$ where F is a primitive function of $x \longmapsto 1 - x$ $(x \in R)$, or $\left(\int x \longmapsto (1 - x)\right)(x)$.

(iii) $\int_a^b f = \left[\int f\right]_a^b$ (or $[F]_a^b$) where $\int f$ (or F) is a primitive function of f.

(iv) $DF(b) = f(b)$, where F satisfies $\int_a^b f = F(b) - F(a)$; that is, F is a primitive function of f. This formula is a statement of the first part of the Fundamental Theorem of Calculus. See Exercise 13.1.2.1. ■

Solution 2

Solution 2

(i)
$$\int_a^b kf = k\int_a^b f$$

becomes

$$\int_a^b x \longmapsto kf(x) = k\int_a^b x \longmapsto f(x),$$

whence

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx.$$

(ii)
$$\int_a^b (f + g) = \int_a^b f + \int_a^b g$$

becomes

$$\int_a^b x \longmapsto (f(x) + g(x)) = \int_a^b x \longmapsto f(x) + \int_a^b x \longmapsto g(x),$$

whence

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

(iii)
$$\int_a^b (f \times Dg) = [f \times g]_a^b - \int_a^b (g \times Df)$$

becomes

$$\int_a^b x \longmapsto f(x)Dg(x) = [x \longmapsto f(x)g(x)]_a^b$$
$$- \int_a^b x \longmapsto g(x)Df(x),$$

whence

$$\int_a^b f(x)\frac{dg(x)}{dx}\,dx = [f(x)g(x)]_a^b - \int_a^b g(x)\frac{df(x)}{dx}\,dx.$$

(iv)
$$\int_a^b (g \circ k) \times Dk = \int_{k(a)}^{k(b)} g$$

becomes

$$\int_a^b x \longmapsto g(k(x))k'(x) = \int_{k(a)}^{k(b)} u \longmapsto g(u),$$

whence

$$\int_a^b g(k(x))\frac{dk(x)}{dx}\,dx = \int_{k(a)}^{k(b)} g(u)\,du.$$

(v)
$$\int_\alpha^\beta g = \int_{h(\alpha)}^{h(\beta)} (g \circ k)Dk$$

becomes

$$\int_\alpha^\beta x \longmapsto g(x) = \int_{h(\alpha)}^{h(\beta)} u \longmapsto g(k(u))Dk(u),$$

whence

$$\int_\alpha^\beta g(x)\,dx = \int_{h(\alpha)}^{h(\beta)} g(k(u))\frac{dk(u)}{du}\,du.$$

■

13.2.8 Appendix (see section 13.2.1)

13.2.8

Appendix

To deduce Equation (2) from Equation (4) we can proceed in this way: let F be any primitive function of the given function f in Equation (2); then from the Fundamental Theorem, kF is a primitive function of kf, and so we have:

$$\int_a^b kf = [kF]_a^b = kF(b) - kF(a) = k[F]_a^b = k\int_a^b f$$

which proves Equation (2).

To reverse the argument and prove Equation (4) from Equation (2), let f be the derived function of the given F in Equation (4); then the Fundamental Theorem gives

$$\int_a^b f = F(b) - F(a),$$

and so Equation (2) gives

$$\int_a^b kf = k\int_a^b f = kF(b) - kF(a);$$

that is, kF is a primitive function of kf, and so, by the Fundamental Theorem again, we have

$$D(kF) = kf = k\,DF,$$

which proves Equation (4). The proofs of Equation (1) from Equation (3) and of Equation (3) from Equation (1) are similar. You may like to try writing them out for yourself.

13.3 REFERENCE TABLES 13.3

13.3.1 Some Standard Primitive Functions 13.3.1

$DF(x)$	$F(x)$
x^{α} $(x \in R)$ with $\alpha \in Z^{+}$	$\dfrac{x^{\alpha+1}}{\alpha+1}$
x^{α} $(x \in R^{+})$ with $\alpha \in R$ and $\alpha \neq -1$	$\dfrac{x^{\alpha+1}}{\alpha+1}$
x^{-1} $(x \in R^{+})$	$\ln x$
$\dfrac{1}{ax+b}$ $(x \in R$ and $ax+b>0)$ with $a \neq 0$	$\dfrac{1}{a}\ln(ax+b)$
$\sin(ax)$ $(x \in R)$, with $a \neq 0$	$\dfrac{-1}{a}\cos(ax)$
$\cos(ax)$ $(x \in R)$, with $a \neq 0$	$\dfrac{1}{a}\sin(ax)$
$\exp(ax)$ $(x \in R)$, with $a \neq 0$	$\dfrac{1}{a}\exp(ax)$

13.3.2 Summary of Rules of Integration

13.3.2

Fundamental Theorem of Calculus

$$\int_a^b DF = [F]_a^b = F(b) - F(a).$$

2 Constant factor rule

$$\int_a^b kf = k\int_a^b f.$$

3 Sum rule

$$\int_a^b f + g = \int_a^b f + \int_a^b g.$$

Integration by parts

$$\int_a^b f \times Dg = [f \times g]_a^b - \int_a^b g \times Df,$$

that is,

$$\int_a^b x \longmapsto f(x) \times Dg(x)$$

$$= [x \longmapsto f(x) \times g(x)]_a^b - \int_a^b x \longmapsto g(x) \times Df(x).$$

Integration by substitution

$$\int_a^b (g \circ k) \times Dk = \int_{k(a)}^{k(b)} g$$

or, alternatively:

$$\int_\alpha^\beta g = \int_{h(\alpha)}^{h(\beta)} (g \circ k) \times Dk,$$

that is,

$$\int_\alpha^\beta x \longmapsto g(x) = \int_{h(\alpha)}^{h(\beta)} u \longmapsto g(k(u)) \times Dk(u),$$

where h is the inverse of the function k.

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1		Functions
2		Errors and Accuracy
3		Operations and Morphisms
4		Finite Differences
5	NO TEXT	
6		Inequalities
7		Sequences and Limits I
8		Computing I
9		Integration I
10	NO TEXT	
11		Logic I — Boolean Algebra
12		Differentiation I
13		Integration II
14		Sequences and Limits II
15		Differentiation II
16		Probability and Statistics I
17		Logic II — Proof
18		Probability and Statistics II
19		Relations
20		Computing II
21		Probability and Statistics III
22		Linear Algebra I
23		Linear Algebra II
24		Differential Equations I
25	NO TEXT	
26		Linear Algebra III
27		Complex Numbers I
28		Linear Algebra IV
29		Complex Numbers II
30		Groups I
31		Differential Equations II
32	NO TEXT	
33		Groups II
34		Number Systems
35		Topology
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